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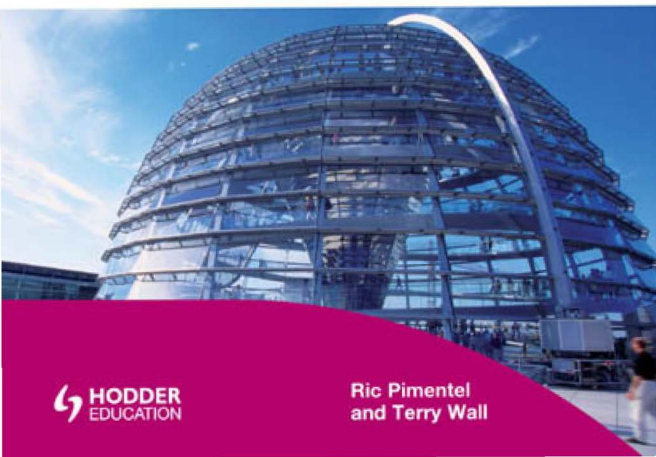
IGCSE International Mathematics



HODDER
EDUCATION

Ric Pimentel
and Terry Wall

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Acknowledgements
p.472–3 Growth charts developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Heath Promotion (2000)

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Contents

Teacher introduction

The new Cambridge IGCSE International Mathematics (0607) syllabus is innovative in that it focuses on the use of the graphics calculator as an integral part of the course. This textbook begins with an initial topic (chapter) which explains very clearly, with photographs of the calculator and of the relevant keys, how one is used. Further explanation, in detail, is then given in particular topics where the graphics calculator is used.

Topics 1–11 cover the Cambridge IGCSE International Mathematics (0607) syllabus. The syllabus headings such as Number, Algebra and Geometry are mirrored in the textbook. Each topic is divided into a number of sections with each section containing its own discrete exercises and student assessments. Teachers may select the order in which these are taught.

Parts of the syllabus are referred to as 'assumed knowledge'. These sections are only included in this textbook if they are a direct base for other work in that section.

The syllabus has an examination paper which tests student understanding of investigations and modelling in mathematics. Where applicable, the textbook includes a number of such questions at the end of a topic, often related to it, so that students can develop their skills in this area throughout the course. (Note that investigations may test earlier learning.) Teachers should encourage students to open and extend these investigations by setting aside dedicated time for the tasks.

Each topic includes a large number of exercises. Teachers may choose to use all or some of these exercises, or to be selective within an exercise depending upon the strengths and weaknesses of their students. In most cases, worked examples and solutions to exercises follow the syllabus/paper system for rounding, i.e. numbers to 3 s.f. and angles in degrees to 1 d.p., unless otherwise stated.

Each topic also provides a number of student assessments which review and reinforce previous work.

On a few occasions, where suitable within the context of the topic, the textbook goes beyond the syllabus requirements. All such instances have been clearly highlighted as Extension material and can be included or excluded.

The digital material which accompanies this textbook can be found online at www.hodderplus.co.uk. It consists of a wealth of valuable resources such as teacher PowerPoint presentations, audio-visual worked examples (Personal Tutors), further sample examination papers, and additional student assessments to be used for homework or class tests. Printable copies of figures from Exercises 5.8–5.11, 5.14, 5.16–5.18, 5.21 and 7.9 are also provided.

This textbook has been written by two experienced mathematics teachers. The authors have written the book to help prepare students thoroughly for the examination and also to encourage them to want to learn more mathematics, perhaps at a higher level.

Terry Wall and Ric Pimentel

Student introduction

The title of this textbook emphasises the internationality of mathematics. A mathematician in Africa may be working with another in Japan to extend work done by a Brazilian in the USA.

Art, music, language and literature are specific to the culture of the country of origin. Opera is European. Noh plays are Japanese. It is not likely that people from different cultures could work together on a piece of Indian art for example.

However, all people in all cultures have tried to understand their world, and mathematics has been a common way of furthering that understanding, even in cultures which have left no written records.

The Ishango Bone from Stone-Age Africa has marks suggesting it was a tally stick. It was the start of arithmetic. 4500 years ago in Samaria (modern day Iraq and Iran), clay tablets show multiplication and division problems.

3600 years ago what is now called 'The Rhind Papyrus' was found in Egypt. It shows simple algebra and fractions. Another, 'The Moscow Papyrus' shows how to find the volume of a truncated pyramid. The Egyptians advanced our knowledge of geometry. The Babylonians worked with arithmetic.

3000 years ago in India the great wise men advanced mathematics and their knowledge travelled to Egypt and later to Greece, then to Europe when great Arab mathematicians took their knowledge with them to Spain. Europe and later America dominated mathematical discoveries from the fifteenth until the twentieth century. It is likely, with the re-emergence of China and India as major world powers, that these countries will again provide great mathematicians and the cycle will be completed.

So when you are studying from this textbook try to be aware that you are following in the footsteps of earlier mathematicians who were excited by the discoveries they had made. These discoveries changed our world.

You may find some of the questions in this book difficult. It is easy when this happens to ask your teacher for help. Remember though that mathematics is intended to stretch the mind. If you are trying to get physically fit you do not stop as soon as things get hard. It is the same with mental fitness. Think logically, try harder. You can solve that difficult problem and get the feeling of satisfaction that comes with learning something new.

Terry Wall and Ric Pimentel

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Introduction to the graphics calculator

Sections

SECTION 1 The history of the calculator

There are many different types of calculators available today. These include basic calculators, scientific calculators and the latest graphics calculators. The history of the calculator is a long one.

The abacus was invented between 2300 BC and 500 BC. It was used mainly for addition and subtraction and is still widely used in parts of South East Asia.

The slide rule was invented in 1621. It was able to do more complex operations than the abacus and continued to be widely used into the early 1970s.

The first mechanical calculator was invented by Blaise Pascal in 1642. It used a system of gears.

The first handheld calculator appeared in 1967 as a result of the development of the integrated circuit.

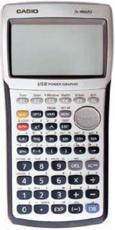
an early abacus

an early slide rule

an early calculator

SECTION 2 The graphics calculator

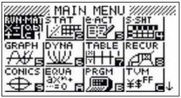
Graphics calculators are a powerful tool used for the study of mathematics in the modern classroom. However, as with all tools, their effectiveness is only apparent when used properly. This section will look at some of the key features of the graphics calculator, so that you start to understand some of its potential. More detailed exploration of its capabilities is integrated into the relevant sections throughout this book. The two models used are the Casio *fx*-9860G and the Texas TI-84 Plus. Many graphics calculators have similar capabilities to the ones shown. However, if your calculator is different, it is important that you take the time to familiarise yourself with it.



Casio

Texas

Here is the home screen (menu/applications) for both calculators.

Casio	**Texas**
 The modes are selected by using the arrows key and then pressing EXE, or by typing the number/letter that appears in the bottom right-hand corner of each square representing a mode. Brief descriptions of the seven most relevant modes are given below.	 The main features are accessed by pressing the appropriate key. Some are explained below.
1 RUN.MAT is used for arithmetic calculations.	 is used to access numerical operations.
2 STAT is used for statistical calculations and for drawing graphs.	 is used for statistical calculations and for drawing graphs of the data entered.
3 S.SHT is a spreadsheet and can be used for calculations and graphs.	 is used for entering the equations of graphs.

4 GRAPH is used for entering the equations of graphs and plotting them.	 is used for graphing functions.
5 DYNA is a dynamic graph mode that enables a family of curves to be graphed simultaneously.	
6 TABLE is used to generate a table of results from an equation.	
7 EQUA is used to solve different types of equations.	

Basic calculations

The aim of the following exercise is to familiarise you with some of the buttons dealing with basic mathematical operations on your calculator. It is assumed that you will already be familiar with the mathematical content.

Exercise 1 Using your graphics calculator, evaluate the following:

Casio	Texas
	

1. a) 625

b) 324

c) $2\sqrt{8} \times 5\sqrt{2}$

Casio	Texas	
		

2. a) $\sqrt[3]{1728}$

b) $\sqrt[4]{1296}$

c) $\sqrt[5]{3125}$

Casio	Texas
	

3. a) 13^2

b) $8^2 \div 4^2$

c) $\sqrt{5^2 + 12^2}$

Casio	Texas
	

4. a) 6^3

b) $9^4 \div 27^2$

c) $\sqrt[4]{\frac{4^3 \times 2^8}{8^2}}$

Casio	Texas	
		

5. a) $(2.3 \times 10^3) + (12.1 \times 10^2)$

b) $(4.03 \times 10^3) + (15.6 \times 10^4) - (1.05 \times 10^4)$

c) $\frac{13.95 \times 10^6}{15.5 \times 10^3} - (9 \times 10^2)$

Graphics calculators also have a large number of memory channels. Use these to store answers which are needed for subsequent calculations. This will minimise rounding errors.

Casio	Texas
 followed by a letter of the alphabet	 followed by a letter of the alphabet

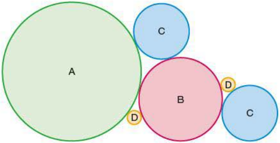
Exercise 2

1. In the following expressions, $a = 5$, $b = 4$ and $c = 2$. Enter each of these values in memory channels A, B and C, respectively, of your calculator and evaluate the following:

a) $a + b + c$ b) $a - (b + c)$ c) $(a + b)^2 - c$

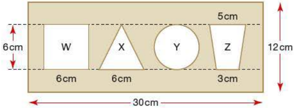
d) $\frac{2(b + c)^3}{(a - c)}$ e) $\frac{4\ \ a^2 - b^2}{c}$ f) $\frac{(ac)^2 + ba^2}{a + b + c}$

2. Circles A, B, C and D have radii 10 cm, 6 cm, 4 cm and 1 cm respectively.



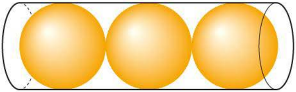
a) Calculate the area of circle A and store your answer in memory channel A.
b) Calculate the area of circle B and store your answer in memory channel B.
c) Calculate the area of each of the circles C and D, storing the answers in memory channels C and D respectively.
d) Using your calculator, evaluate A + B + 2C + 2D.
e) What does the answer to Q.2(d) represent?

3. The diagram shows a child's shape-sorting toy. The top consists of a rectangular piece of wood of dimension 30 cm × 12 cm. Four shapes W, X, Y and Z are cut out of it.



a) Calculate the area of the triangle X. Store the answer in your calculator's memory.
b) Calculate the area of the trapezium Z. Store the answer in your calculator's memory.
c) Calculate the total area of the shapes W, X, Y and Z.
d) Calculate the area of the rectangular piece of wood left once the shapes have been cut out.

4. Three balls just fit inside a cylindrical tube as shown. The radius (r) of each ball is 5 cm.



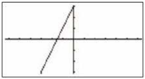
a) Use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, to calculate the volume of one of the balls. Store the answer in the memory of your calculator.
b) Calculate the volume of the cylinder.
c) Calculate the volume of the cylinder **not** occupied by the three balls.

SECTION 3 Plotting graphs

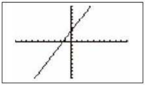
One of a graphics calculator's principal features is to plot graphs of functions. This helps to visualise what the function looks like and, later on, will help solve a number of different types of problems. This section aims to show how to graph a variety of different functions.

For example, to plot a graph of the function $y = 2x + 3$, use the following buttons on your calculator:

Casio



Texas



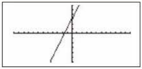
It may be necessary to change the scale on the axes in order to change how much of the graph, or what part of the graph, can be seen. This can be done in several ways, two of which are described here:

- by using the zoom facility

Casio

to zoom out, or

to zoom in.

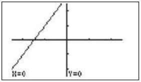


It is possible to reposition the graph by using the  key

Texas

to zoom in, or

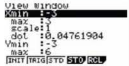
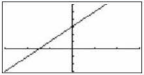
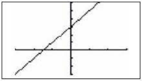
to zoom out.



It is possible to reposition the centre of enlargement by using the

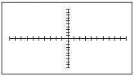
keys before pressing 

- by changing the scale manually.

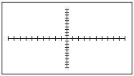
<table>
<tr><th colspan="3">Casio</th></tr>
<tr>
<td>
 after values are entered</td>
<td>View Window
Xmin :-3
max :3
scale:1
dot :0.04761904
Ymin :-3
max :6
 </td>
<td></td>
</tr>
<tr><td colspan="3">Xmin: minimum value on the <i>x</i>-axis, Xmax: maximum value on the <i>x</i>-axis,
Xscale: spacing of the <i>x</i>-axis increments, Xdot: value that relates to one <i>x</i>-axis dot (this is set automatically).</td></tr>
<tr><th colspan="3">Texas</th></tr>
<tr>
<td>
 after values are entered</td>
<td>WINDOW
Xmin=-3
Xmax=3
Xscl=1
Ymin=-3
Ymax=6
Yscl=1
Xres=1</td>
<td></td>
</tr>
<tr><td colspan="3">Xmin: minimum value on the <i>x</i>-axis, Xmax: maximum value on the <i>x</i>-axis,
Xscl: spacing of the <i>x</i>-axis increments.</td></tr>
</table>

Exercise 3 In Q.1–4 the axes have been set to their default settings, i.e. Xmin = –10, Xmax = 10, Xscale = 1, Ymin = –10, Ymax = 10, Yscale = 1.

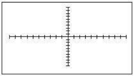
i)



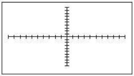
ii)



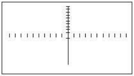
iii)



iv)

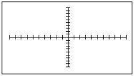


v)

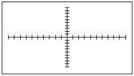


vi)

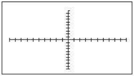
vii)



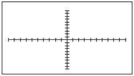
viii)



ix)



x)



1. By using your graphics calculator to graph the following functions, match each of them to the correct graph.

a) $y = 2x + 6$

b) $y = \frac{1}{2}x - 2$

c) $y = -x + 5$

d) $y = -\frac{5}{x}$

e) $y = x^2 - 6$

f) $y = (x - 4)^2$

g) $y = -(x + 4)^2 + 4$

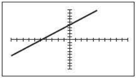
h) $y = \frac{1}{2}(x + 3)^3$

i) $y = -\frac{1}{3}x^3 + 2x - 1$

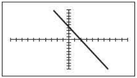
j) $y = -6$

2. In each of the following, a function and its graph is given. Using your graphics calculator, enter a function that produces a reflection of the original function in the x-axis.

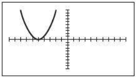
a) $y = x + 5$



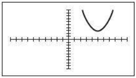
b) $y = -2x + 4$



c) $y = (x + 5)^2$



d) $y = (x - 5)^2 + 3$



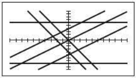
3. Using your graphics calculator, enter a function that produces a reflection in the y-axis of each of the original functions in Q.2.

4. By entering appropriate functions into your graphics calculator:

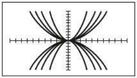
i) make your graphical calculator screen look like the ones shown

ii) write down the functions you used.

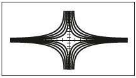
a)



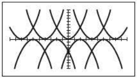
b)



c)



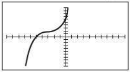
d)



Intersections

When graphing a function it is often necessary to find where it intersects one or both of the axes. If more than one function is graphed on the same axes, it may also be necessary to find where the graphs intersect each other. Graphics calculators can be used to find the coordinates of any points of intersection.

Worked example



Find where the graph of $y = \frac{1}{5}(x + 3)^3 + 2$ intersects both the x- and y-axes.

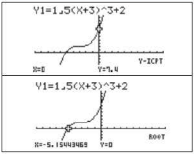
The graph shows that $y = \frac{1}{5}(x + 3)^3 + 2$ intersects each of the axes once.

To find the coordinates of the points of intersection:

Casio

SHIFT G-Solv F5 Sketch F4 to find the y-intercept.

SHIFT G-Solv F5 Trace F1 to find the root (the x-intercept).



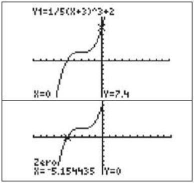
Y1=1⌟5(X+3)^3+2

X=0 Y=7.4 Y-ICPT

Y1=1⌟5(X+3)^3+2

X=-5.15443469 Y=0 ROOT

Note: The **root** of an equation refers to the point at which it crosses the x-axis.
The point at which the equation crosses the y-axis is known as the **y-intercept**.

Texas	
2ND CALC TRACE 1 then enter $x = 0$ to find the y-intercept. 2ND CALC TRACE 2 Use ◀ ▶ to move the cursor to a point to the left of the intersection with the x-axis, then press ENTER. Use ◀ ▶ to move the cursor to a point to the right of the intersection with the x-axis, then press ENTER and ENTER again.	 Y1=1/5(X+3)^3+2 X=0 Y=7.4 Zero X=-5.154435 Y=0
Note: The TI-84 prompts the user to identify a point to the left of the intersection with the x-axis (left bound) and then a point to the right of the intersection (right bound).	

Exercise 4

1. Find an approximate solution to where the following graphs intersect both the x-and y-axes using your graphics calculator.

a) $y = x^2 - 3$

b) $y = (x + 3)^2 + 2$

c) $y = \frac{1}{2}x^3 - 2x^2 + x + 1$

d) $y = \frac{-5}{x + 2} + 6$

2. Find the coordinates of the point(s) of intersection of each of the following pairs of equations using your graphics calculator.

a) $y = x + 3$ and $y = -2x - 2$

b) $y = -x + 1$ and $y = \frac{1}{2}(x^2 - 3)$

c) $y = -x^2 + 1$ and $y = \frac{1}{2}(x^2 - 3)$

d) $y = -\frac{1}{4}x^3 + 2x^2 - 3$ and $y = \frac{1}{2}x^2 - 2$

SECTION 4

Tables of results

A function such as $y = \frac{3}{x} + 2$ implies that there is a relationship between y and x.

To plot the graph manually, the coordinates of several points on the graph need to be calculated and then plotted. Graphics calculators have the facility to produce a table of values giving the coordinates of some of the points on the line.

Worked example For the function $y = \frac{3}{x} + 2$, complete the following table of values using the table facility of your graphics calculator:

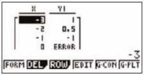
x	–3	–2	–1	0	1	2	3
y							

Casio

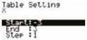
Enter function $y = \frac{3}{x} + 2$

to enter the values of x to appear in the table.

to display the table.



```
      X      Y1
    [ -3      1 ]
    | -2    0.5 |
    | -1     -1 |
    [  0  ERROR ]
                    -3
```

The screen  shows that the x-values range from –3 to 3 in increments of 1.

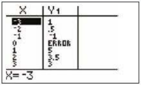
```
Table Setting
X
Start:-3
End  :3
Step :1
```

Once the table is displayed, the remaining results can be viewed by using .

Texas

Enter function $y = \frac{3}{x} + 2$

to enter the values of x and display the table.



```
   X    | Y1    |
--------+-------+------
  -3    | 1     |
  -2    | .5    |
  -1    | -1    |
  0     | ERROR |
  1     | 5     |
  2     | 3.5   |
  3     | 3     |
--------+-------+------
X= -3
```

```
TABLE SETUP
 TblStart=-3
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

The screen shows that the *x*-values start at –3 and increase in increments of 1.

Once the table is displayed, further results can be viewed by using .

Exercise 5

1. Copy and complete the tables of values for the following functions using the table facility of your graphics calculator:

a) $y = x^2 + x - 4$

x	–3	–2	–1	0	1	2	3
y							

b) $y = x^3 + x^2 - 10$

x	–3	–2	–1	0	1	2	3
y							

c) $y = \frac{4}{x}$

x	0	0.5	1	1.5	2	2.5	3
y							

d) $y = \sqrt{(x+1)}$

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3
y									

2. A car accelerates from rest. Its speed, y m/s, x seconds after starting, is given by the equation $y = 1.8x$.

a) Using the table facility of your graphics calculator, calculate the speed of the car every 2 seconds for the first 20 seconds.

b) How fast is the car travelling after 10 seconds?

3. A ball is thrown vertically upwards. Its height, y metres, x seconds after launch is given by the equation $y = 15x - 5x^2$.

a) Using the table facility of your graphics calculator, calculate the height of the ball each $\frac{1}{2}$ second during the first 4 seconds.

b) What is the greatest height reached by the ball?

c) How many seconds after its launch did the ball reach its highest point?

d) After how many seconds did the ball hit the ground?

e) In the context of this question, why can the values for $x = 3.5$ and 4 be ignored?

SECTION 5 Lists

Data is often collected and then analysed so that observations and conclusions can be made. Graphics calculators have the facility for storing data as lists. Once stored as a list, many different types of calculations can be carried out. This section will explain how to enter data as a list and then how to carry out some simple calculations.

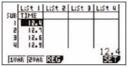
Worked examples **a)** An athlete records her time (seconds) in ten races for running 100 m. These are shown below:

12.4 12.7 12.6 12.9 12.4 12.3 12.7 12.4 12.5 13.1

Calculate the mean, median and mode for this set of data using the list facility of your graphics calculator.

Casio

Enter the data in List 1.



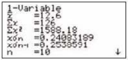
```
 |List 1|List 2|List 3|List 4|
SUB|TIME  |      |      |      |
 1 |  12.4|      |      |      |
 2 |  12.7|      |      |      |
 3 |  12.6|      |      |      |
 4 |  12.9|      |      |      |
                           12.4
 1VAR 2VAR REG              SET
```

to specify which list the data is in and its frequency.

```
1Var XList   :List1
1Var Freq    :1
2Var XList   :List1
2Var YList   :List2
2Var Freq    :1

LIST
```

to scroll through the full list.

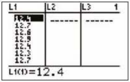


```
1-Variable
x̄      =12.6
Σx     =126
Σx²    =1588.18
xσn    =0.24083189
xσn-1  =0.2538591
n      =10           ↓
```

The screen displays various statistical measures.
$\overline{x}$ is the mean, n is the number of data items, Med is the median, Mod is the modal value, Mod: F is the frequency of the modal values.

Texas

Enter the data in List 1.

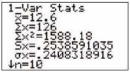


```
L1      L2      L3      1
12.4    ------  ------
12.7
12.6
12.9
12.4
12.3
12.7
L1(1)=12.4
```

to apply calculations to the data in List 1.

```
1-Var Stats L1
```

to scroll through the full list.



```
1-Var Stats
 x̄=12.6
 Σx=126
 Σx²=1588.18
 Sx=.2538591035
 σx=.2408318916
↓n=10
```

The screen displays various statistical measures.
$\bar{x}$ is the mean, *n* is the number of data items, Med is the median. The T1-84 does not display the modal value.

If a lot of data is collected, it is often presented in a frequency table.

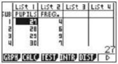
b) The numbers of students in 30 Maths classes are shown in the frequency table below:

Number of students	Frequency
27	4
28	6
29	9
30	7
31	3
32	1

Calculate the mean, median and mode for this set of data using the list facility of your graphics calculator.

Casio

Enter the number of students in List 1 and the frequency in List 2.



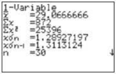
```
 |List 1|List 2|List 3|List 4
SUB|PUPILS|FREQ. |      |
 1 |    27|     4|      |
 2 |    28|     6|      |
 3 |    29|     9|      |
 4 |    30|     7|      |
                         27
GRPH CALC TEST INTR DIST ▷
```

to specify which lists the data and the frequency are in.

```
1Var XList  :List1
1Var Freq   :List2
2Var XList  :List1
2Var YList  :List2
2Var Freq   :1

 1  LIST
```

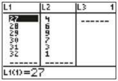
to scroll through the full list.



```
1-Variable
x̄     =29.0666666
Σx    =872
Σx²   =25396
xσn   =1.28927197
xσn-1 =1.3113124
n     =30          ↓
```

Texas

Enter the number of students in List 1 and the frequency in List 2.

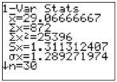


```
L1     |L2     |L3     1
 27    | 4     | ------
 28    | 6     |
 29    | 9     |
 30    | 7     |
 31    | 3     |
 32    | 1     |
 ------| ------|
L1(1)=27
```

to specify that the data is in L1 and the frequency in L2.

```
1-Var Stats L1,L
2█
```

to scroll through the full set of results.



```
1-Var Stats
 x̄=29.06666667
 Σx=872
 Σx²=25396
 Sx=1.311312407
 σx=1.289271974
↓n=30
```

Exercise 6

1. Find the mean, the median and, if possible, the mode of these sets of numbers using the list facility of your graphics calculator.
   a) 3, 6, 10, 8, 9, 10, 12, 4, 6, 10, 9, 4
   b) 12.5, 13.6, 12.2, 14.4, 17.1, 14.8, 20.9, 12.2

2. During a board game a player makes a note of the numbers he rolls on the dice. These are shown in the frequency table below:

Number on dice	1	2	3	4	5	6
Frequency	3	8	5	2	5	7

   Find the mean, the median and, if possible, the modal dice roll using the list facility of your graphics calculator.

3. A class of 30 students sat two Maths tests. Their scores out of 10 are recorded in the frequency tables below:

Test A										
Score	1	2	3	4	5	6	7	8	9	10
Frequency	3	2	4	3	1	8	3	1	3	2

Test B										
Score	1	2	3	4	5	6	7	8	9	10
Frequency	4	1	0	0	0	24	0	0	0	1

   a) Find the mean, the median and, if possible, the mode for each test using the list facility of your graphics calculator.
   b) Comment on any similarities or differences in your answers to Q.3(a).
   c) Which test did the class find easiest? Give reasons for your answer.

TOPIC 1

Number

This topic will cover the following syllabus content:

1.6 Absolute value $|x|$

1.8 Percentages including applications such as interest and profit

1.9 Meaning of exponents (powers, indices) in $\mathbb{Q}$
Standard form $a \times 10^n$ where $1 \leq a < 10$ and $n \in \mathbb{Z}$
Rules for exponents

1.10 Surds (radicals), simplification of square root expressions
Rationalisation of the denominator

1.13 Speed, distance, time problems

Sections

SECTION 1 Hindu mathematicians

Aryabatta (476–550)

In 1300BC a Hindu teacher named Laghada used geometry and trigonometry for his astronomical calculations. At around this time, other Indian mathematicians solved quadratic and simultaneous equations.

Much later, in about AD500, another Indian teacher, Aryabatta, worked on approximations for pi and on the trigonometry of the sphere. He realised that not only did the planets go round the Sun but that their paths were elliptic.

Brahmagupta, a Hindu, was the first to treat zero as a number in its own right. This helped to develop the decimal system of numbers.

One of the greatest mathematicians of all time was Bhascara who, in the twelfth century, worked in algebra and trigonometry. He discovered that:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

His work was taken to Arabia and later to Europe.

SECTION 2 Vocabulary for sets of numbers

■ Natural numbers

A child learns to count 'one, two, three, four, …'. These are sometimes called the counting numbers or whole numbers.

The child will say 'I am three', or 'I live at number 73'.

If we include the number 0, then we have the set of numbers called the **natural numbers**.

The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.

■ Integers

On a cold day, the temperature may be 4 °C at 10p.m. If the temperature drops by a further 6 °C, then the temperature is 'below zero'; it is –2 °C.

If you are overdrawn at the bank by £200, this might be shown as –£200.

The set of **integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}$ is therefore an extension of $\mathbb{N}$. Every natural number is an integer.

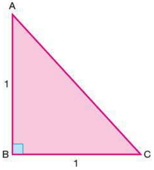
Rational numbers

A child may say 'I am three'; she may also say 'I am three and a half', or even 'three and a quarter'. $3\frac{1}{2}$ and $3\frac{1}{4}$ are **rational numbers**. All rational numbers can be written as a fraction whose denominator is not zero. All terminating and recurring decimals are rational numbers as they can also be written as fractions, e.g.

$$0.2 = \tfrac{1}{5} \quad 0.3 = \tfrac{3}{10} \quad 7 = \tfrac{7}{1} \quad 1.53 = \tfrac{153}{100} \quad 0.\dot{2} = \tfrac{2}{9}$$

The set of rational numbers $\mathbb{Q}$ is an extension of the set of integers.

Real numbers



Numbers which cannot be expressed as a fraction are not rational numbers; they are **irrational numbers**.

Using Pythagoras' rule in the diagram to the left, the length of the hypotenuse AC is found as:

$$\begin{aligned} AC^2 &= 1^2 + 1^2 \\ AC^2 &= 2 \\ AC &= \ \ 2 \end{aligned}$$

$\sqrt{2} = 1.41421356\ldots$. The digits in this number do not recur or repeat. This is a property of all irrational numbers. Another example of an irrational number you will come across is π (pi). It is the ratio of the circumference of a circle to the length of its diameter. Although it is often rounded to 3.14, the digits continue indefinitely never repeating themselves.

The set of rational and irrational numbers together form the set of **real numbers** $\mathbb{R}$.

Absolute numbers

The absolute value of a number refers to its magnitude and is therefore not affected by whether the number is positive or negative. Therefore the absolute value of –3 is 3. The absolute value of 3 is also 3.

To indicate that the absolute value of a number is needed, the notation $|x|$ is used:

$$|-3| = 3$$
and $$|3| = 3$$

The absolute value and its application to functions is covered in Topic 3. Its application to vectors is covered in Topic 5.

SECTION 3

Surds

The roots of some numbers produce rational answers, for example:

$$\sqrt{16} = 4 \qquad \sqrt[5]{32} = 2 \qquad \sqrt{\frac{4}{9}} = \frac{2}{3}$$

If roots cannot be written as rational numbers, they are known as **surds**. Surds are therefore irrational numbers. Examples of surds include $\sqrt{2}$, $\sqrt{3}$ and $\sqrt[4]{10}$.

If an answer to a question is a surd, then leaving the answer in surd form is exact. Using a calculator to calculate a decimal equivalent is only an approximation.

In the Pythagoras example above, the length of the hypotenuse was given as 2. This is the exact length. A calculator will state that $\sqrt{2} = 1.414213562$, but this is only an approximation correct to nine decimal places.

You should always leave answers in exact form unless you are asked to give your answer to a certain number of decimal places.

Simplification of surds

If $\sqrt{x}$ cannot be simplified further then it is in basic form. 17 and 43 are prime numbers so $\sqrt{17}$ and $\sqrt{43}$ cannot be simplified further. The square root of some numbers which are not prime such as $\sqrt{20}$, $\sqrt{63}$ and $\sqrt{363}$ can be simplified:

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

$$\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$$

$$\sqrt{363} = \sqrt{121 \times 3} = \sqrt{121} \times \sqrt{3} = 11\sqrt{3}$$

Note: Each time the original number is written as the product of two numbers, one of which is square.

Surds can be manipulated and simplified according to a number of rules. These are:

Rule	Example
$\sqrt{a} \times \sqrt{a} = a$	$\sqrt{3} \times \sqrt{3} = 3$
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$	$\sqrt{3} \times \sqrt{5} = \sqrt{15}$
$\frac{a}{b} = \sqrt{\frac{a}{b}}$	$\frac{8}{2} = \sqrt{\frac{8}{2}} = \quad 4 = 2$
$a \quad b \times \quad c = a \quad bc$	$3 \quad 5 \times \quad 6 = 3 \quad 30$
$a + \quad b \neq \quad a + b$	$4 + \quad 9 \neq \quad 13$ as $2 + 3 \neq \quad 13$

Worked examples

a) Simplify $\sqrt{3} + \sqrt{12}$.

In order to add surds, they must both be multiples of the same surd.

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Therefore $\sqrt{3} + \sqrt{12}$ can be written as $\sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$.

b) Expand and simplify $(2 + \sqrt{3})\ (3 - \sqrt{3})$.

Multiplying both terms in the first bracket by both terms in the second bracket gives: $2 \times 3 - 2\sqrt{3} + 3\sqrt{3} - \sqrt{3} \times \sqrt{3}$

$\Rightarrow 6 + \sqrt{3} - 3$

$\Rightarrow 3 + \sqrt{3}$

Exercise 1.1

1. Simplify the following surds:

a) $\sqrt{24}$ b) $\sqrt{48}$ c) $\sqrt{75}$

d) $\sqrt{700}$ e) $\sqrt{162}$

2. Simplify the following where possible:

a) $\sqrt{47}$ b) $\sqrt{98}$ c) $\sqrt{8}$

d) $\sqrt{51}$ e) 432

3. Simplify these expressions:

a) $\sqrt{3} \times \sqrt{3}$ b) $\sqrt{5} \times \sqrt{5}$ c) $\sqrt{3} + \sqrt{3}$

d) $\sqrt{2} + \sqrt{2}$ e) $3\sqrt{5} - \sqrt{5}$ f) $4\sqrt{7} + 3\sqrt{7}$

4. Simplify these expressions:

a) $\sqrt{2} + \sqrt{8}$ b) $\sqrt{7} + \sqrt{63}$ c) $\sqrt{20} + \sqrt{45}$

d) $3\sqrt{2} - 4\sqrt{8}$ e) $5\sqrt{10} - \sqrt{40}$ f) $\sqrt{28} - \sqrt{7}$

5. Expand the following expressions and simplify as far as possible:

a) $(3 + \sqrt{2})(1 + \sqrt{2})$ b) $(2 - \sqrt{2})(3 + \sqrt{2})$

c) $(5 + \sqrt{5})(3 - \sqrt{5})$ d) $(1 + 2\sqrt{3})(4 - 3\sqrt{3})$

e) $(3 + 3\sqrt{2})(5 - 2\sqrt{2})$ f) $(3 - 2\sqrt{5})(4 - 3\sqrt{5})$

Rationalising fractions with surds in the denominator

$\dfrac{1}{2}$ is a fraction with a surd in the denominator.

It is considered mathematically more elegant if fractions are written without surds in their denominator. Removing surds from the denominator of a fraction is known as rationalising the denominator.

To rationalise, the fraction must be multiplied by a fraction that is equivalent to 1 but which eliminates the surd.

Worked example Rationalise $\frac{1}{\sqrt{2}}$.

Multiplying the fraction by $\frac{\sqrt{2}}{\sqrt{2}}$ gives:

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Note that $\frac{\sqrt{2}}{\sqrt{2}} = 1$ therefore $\frac{1}{\sqrt{2}}$ is unchanged when multiplied by 1.

In general, to rationalise a fraction of the form $\frac{a}{b}$, multiply by $\frac{b}{\sqrt{b}}$ to give $\frac{a\quad b}{b}$.

Exercise 1.2

1. Rationalise the following fractions, simplifying your answers where possible:

a) $\frac{1}{5}$ b) $\frac{2}{7}$ c) $\frac{2}{2}$

d) $\frac{3}{\sqrt{3}}$ e) $\frac{4}{\sqrt{7}}$ f) $\frac{4}{\sqrt{8}}$

g) $\frac{5}{\sqrt{5}}$ h) $\frac{6}{\sqrt{3}}$ i) $\frac{5}{\sqrt{15}}$

j) $\frac{7}{\sqrt{3}}$ k) $\frac{12}{\sqrt{2}}$

2. Evaluate the following, leaving your answer in simplified and rationalised form:

a) $\frac{3}{\sqrt{2}} + \frac{1}{2}$ b) $\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}}$ c) $\frac{2}{\sqrt{5}} + \frac{\sqrt{5}}{2}$

The denominator is not however always just a single term, e.g. $\frac{1}{3+\sqrt{2}}$.

Rationalising this type of fraction is not just a case of multiplying by $\frac{3+\quad 2}{3+\quad 2}$ as this will not eliminate the surd in the denominator, i.e. $\frac{1}{3+\quad 2} \times \frac{3+\quad 2}{3+\quad 2} = \frac{3+\quad 2}{11+6\quad 2}$.

To rationalise this type of fraction requires an understanding of the difference of two squares, i.e. that $(a-b)(a+b) = a^2 - b^2$. This demonstrates that if either a or b are surds, the result involving a^2 and b^2 will be rational.

Therefore to rationalise $\frac{1}{3+\sqrt{2}}$:

$$\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2} = \frac{3-\sqrt{2}}{7}$$

Worked example Rationalise the denominator of the fraction $\frac{1}{3-2\sqrt{2}}$.

$$\frac{1}{3-2\quad 2} \times \frac{3+2\sqrt{2}}{3+2\quad 2} = \frac{3+2\sqrt{2}}{9+6\quad 2-6\quad 2-4\quad 2\quad 2}$$

$$\Rightarrow \frac{3+2\quad 2}{9-8} = 3+2\sqrt{2}$$

Exercise 1.3

Rationalise the following fractions. Where possible, leave your answers in simplified form.

1. $\frac{1}{\sqrt{2}+1}$ **2.** $\frac{1}{\sqrt{3}-1}$ **3.** $\frac{3}{2-\sqrt{3}}$

4. $\frac{2}{\sqrt{2}-1}$ **5.** $\frac{5}{2+\sqrt{5}}$ **6.** $\frac{7}{1+\sqrt{7}}$

7. $\frac{1}{3-\sqrt{3}}$ **8.** $\frac{2}{2-\sqrt{3}}$ **9.** $\frac{1}{\sqrt{6}-\sqrt{5}}$

10. $\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{2}-1}$

Exercise 1.4

1. State to which of the sets ℕ, ℤ, ℚ or ℝ these numbers belong.

a) 3 b) –5 c) $\sqrt{3}$

d) $11.\dot{3}$

In Q.2–6 state, giving reasons, whether each number is rational or irrational.

2. a) 1.3 b) $0.\dot{6}$ c) $\quad 3$

3. a) $-2\frac{2}{5}$ b) $\quad 25$ c) ${}^{3}\quad 8$

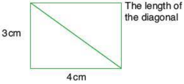
4. a) $\quad 7$ b) 0.625 c) $0.\dot{1}\dot{1}$

5. a) $\sqrt{2} \times \sqrt{3}$ b) $\sqrt{2} + \sqrt{3}$ c) $(\sqrt{2} \times \sqrt{3})^2$

6. a) $\dfrac{\sqrt{8}}{\sqrt{2}}$ b) $\dfrac{2\sqrt{5}}{\sqrt{20}}$ c) $4 + (\sqrt{9} - 4)$

In Q.7–10, state, giving reasons, whether the answer required is a rational or irrational number.

7.



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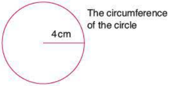

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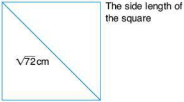

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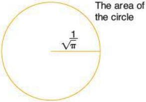


SECTION 4

Percentages

You should already be familiar with the percentage equivalent of simple fractions and decimals as outlined in the table below:

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{3}{4}$	0.75	75%
$\frac{1}{8}$	0.125	12.5%
$\frac{3}{8}$	0.375	37.5%
$\frac{5}{8}$	0.625	62.5%
$\frac{7}{8}$	0.875	87.5%
$\frac{1}{10}$	0.1	10%
$\frac{2}{10}$ or $\frac{1}{5}$	0.2	20%
$\frac{3}{10}$	0.3	30%
$\frac{4}{10}$ or $\frac{2}{5}$	0.4	40%
$\frac{6}{10}$ or $\frac{3}{5}$	0.6	60%
$\frac{7}{10}$	0.7	70%
$\frac{8}{10}$ or $\frac{4}{5}$	0.8	80%
$\frac{9}{10}$	0.9	90%

Simple percentages

Worked examples

a) Of 100 sheep in a field, 88 are ewes.

i) What percentage of the sheep are ewes?
88 out of 100 are ewes
= 88%

ii) What percentage are not ewes?
12 out of 100 are not ewes
= 12%

b) A gymnast scored marks out of 10 from five judges. The marks awarded were: 8.0, 8.2, 7.9, 8.3, 7.6. Express these marks as percentages.

$\frac{8.0}{10} = \frac{80}{100} = 80\%$ $\frac{8.2}{10} = \frac{82}{100} = 82\%$ $\frac{7.9}{10} = \frac{79}{100} = 79\%$

$\frac{8.3}{10} = \frac{83}{100} = 83\%$ $\frac{7.6}{10} = \frac{76}{100} = 76\%$

c) Convert the following percentages into fractions and decimals:

i) 27% ii) 5%

$\frac{27}{100} = 0.27$ $\frac{5}{100} = 0.05$

Exercise 1.5

1. In a survey of 100 cars, 47 were white, 23 were blue and 30 were red. Express each of these numbers as a percentage of the total.

2. $\frac{7}{10}$ of the surface of the Earth is water. Express this as a percentage.

3. There are 200 birds in a flock. 120 of them are female. What percentage of the flock are:

a) female? b) male?

4. Write these percentages as fractions of 100:

a) 73% b) 28%
c) 10% d) 25%

5. Write these fractions as percentages:

a) $\frac{27}{100}$ b) $\frac{3}{10}$
c) $\frac{7}{50}$ d) $\frac{1}{4}$

6. Convert the following percentages to decimals:

a) 39% b) 47% c) 83%
d) 7% e) 2% f) 20%

7. Convert the following decimals to percentages:

a) 0.31 b) 0.67 c) 0.09
d) 0.05 e) 0.2 f) 0.75

Calculating a percentage of a quantity

Worked examples

a) Find 25% of 300 m.

25% can be written as 0.25.
$0.25 \times 300\,\text{m} = 75\,\text{m}$

b) Find 35% of 280 m.

35% can be written as 0.35.
$0.35 \times 280\,\text{m} = 98\,\text{m}$

Exercise 1.6

1. Write the percentage equivalent of the following fractions:

a) $\frac{1}{4}$ b) $\frac{2}{3}$ c) $\frac{5}{8}$
d) $1\frac{4}{5}$ e) $4\frac{9}{10}$ f) $3\frac{7}{8}$

2. Write the decimal equivalent of the following:

a) $\frac{3}{4}$ b) 80% c) $\frac{1}{5}$
d) 7% e) $1\frac{7}{8}$ f) $\frac{1}{6}$

3. Evaluate the following:

a) 25% of 80 b) 80% of 125 c) 62.5% of 80
d) 30% of 120 e) 90% of 5 f) 25% of 30

4. Evaluate the following:

a) 17% of 50 b) 50% of 17 c) 65% of 80
d) 80% of 65 e) 7% of 250 f) 250% of 7

5. In a class of 30 students, 20% have black hair, 10% have blonde hair and 70% have brown hair. Calculate the number of students with:

a) black hair b) blonde hair c) brown hair.

6. A survey conducted among 120 children looked at which type of meat they preferred. 55% said they preferred beef, 20% said they preferred chicken, 15% preferred lamb and 10% liked none of these. Calculate the number of children in each category.

7. A survey was carried out in a school to see what nationality its students were. Of the 220 students in the school, 65% were English, 20% were Pakistani, 5% were Greek and 10% belonged to other nationalities. Calculate the number of students of each nationality.

8. A shopkeeper keeps a record of the number of items he sells in one day. Of the 150 items he sold, 46% were newspapers, 24% were pens, 12% were books whilst the remaining 18% were other assorted items. Calculate the number of each item he sold.

Expressing one quantity as a percentage of another

To express one quantity as a percentage of another, write the first quantity as a fraction of the second and then multiply by 100.

Worked example

In an examination a girl obtains 69 marks out of 75. Express this result as a percentage.

$$\frac{69}{75} \times 100\% = 92\%$$

Exercise 1.7

1. Express the first quantity as a percentage of the second.

a) 24 out of 50
b) 46 out of 125
c) 7 out of 20
d) 45 out of 90
e) 9 out of 20
f) 16 out of 40
g) 13 out of 39
h) 20 out of 35

2. A hockey team plays 42 matches. It wins 21, draws 14 and loses the rest. Express each of these results as a percentage of the total number of games played.

3. Four candidates stood in an election:

A received 24 500 votes
B received 18 200 votes
C received 16 300 votes
D received 12 000 votes

Express each of these as a percentage of the total votes cast.

4. A car manufacturer produces 155 000 cars a year. The cars are available for sale in six different colours. The numbers sold of each colour were:

Red 55 000
Blue 48 000
White 27 500
Silver 10 200
Green 9300
Black 5000

Express each of these as a percentage of the total number of cars produced. Give your answers to 3 s.f.

Percentage increases and decreases

Worked examples

a) A garage increases the price of a truck by 12%. If the original price was $14500, calculate its new price.

Note: the original price represents 100%, therefore the increased price can be represented as 112%.

New price = 112% of $14 500
= 1.12 × $14500
= $16240

b) A Saudi doctor has a salary of 16000 Saudi riyals per month. If his salary increases by 8%, calculate:
i) the amount extra he receives a month
ii) his new monthly salary.

i) Increase = 8% of 16 000 riyals
= 0.08 × 16000 riyals = 1280 riyals
ii) New salary = old salary + increase
= 16000 + 1280 riyals per month
= 17280 riyals per month

c) A shop is having a sale. It sells a set of tools costing $130 at a 15% discount. Calculate the sale price of the tools.

Note: The old price represents 100%, therefore the new price can be represented as (100 – 15)% = 85%.

85% of $130 = 0.85 × $130 = $110.50

Exercise 1.8

1. Increase the following by the given percentage:
a) 150 by 25%
b) 230 by 40%
c) 7000 by 2%
d) 70 by 250%
e) 80 by 12.5%
f) 75 by 62%

2. Decrease the following by the given percentage:
a) 120 by 25%
b) 40 by 5%
c) 90 by 90%
d) 1000 by 10%
e) 80 by 37.5%
f) 75 by 42%

3. In each part below, the first number is increased to become the second number. Calculate the percentage increase in each case.
a) $50 \rightarrow 60$
b) $75 \rightarrow 135$
c) $40 \rightarrow 84$
d) $30 \rightarrow 31.5$
e) $18 \rightarrow 33.3$
f) $4 \rightarrow 13$

4. In each part below, the first number is decreased to become the second number. Calculate the percentage decrease in each case.
a) $50 \rightarrow 25$
b) $80 \rightarrow 56$
c) $150 \rightarrow 142.5$
d) $3 \rightarrow 0$
e) $550 \rightarrow 352$
f) $20 \rightarrow 19$

5. A farmer increases the yield on his farm by 15%. If his previous yield was 6500 tonnes, what is his current yield?

6. The cost of a computer in a Brazilian computer store is reduced by 12.5% in a sale. If the computer was priced at 7800 Brazilian real (BRL), what is its price in the sale?

7. A winter coat is priced at £100. In the sale its price is reduced by 25%.
   a) Calculate the sale price of the coat.
   b) After the sale its price is increased by 25% again. Calculate the price of the coat after the sale.

8. A farmer takes 250 chickens to be sold at a market. In the first hour he sells 8% of his chickens. In the second hour he sells 10% of those that were left.
   a) How many chickens has he sold in total?
   b) What percentage of the original number did he sell in the two hours?

9. The number of fish on a fish farm increases by approximately 10% each month. If there were originally 350 fish, calculate to the nearest 100 how many fish there would be after 12 months.

Simple interest

Interest is money added by a bank or building society to sums deposited by customers, or money charged by a bank or building society to customers for borrowing. The money deposited or borrowed is called the **capital**. The **percentage interest** is the given rate and the money is usually left or borrowed for a fixed period of time.

The following formula can be used to calculate **simple interest**:

$$I = \frac{Crn}{100}$$

where I = the simple interest paid
C = the capital (the amount borrowed or lent)
n = number of time periods (usually years)
r = percentage rate

It is easy to understand this formula if we look at using percentages as shown in the example below.

To work out 15% of \$300, simply calculate $\frac{15}{100} \times 300$. If this is repeated 4 times the calculation becomes $\frac{15}{100} \times 300 \times 4$.

This can also be written as $\frac{15 \times 300 \times 4}{100}$.

Therefore to work out r% of C, the calculation is $\frac{r}{100} \times C$ which can be written as $\frac{Cr}{100}$. If this is repeated n times, the calculation is $\frac{Crn}{100}$.

Worked examples

a) Find the simple interest earned on \$250 deposited for six years at 8% p.a.?

$$\begin{aligned} I &= \frac{Crn}{100} \\ &= \frac{250 \times 8 \times 6}{100} \\ &= 120 \end{aligned}$$

The interest paid is \$120.

b) How long will it take for a sum of €250 invested at 8% p.a. to earn interest of €80?

$$\begin{aligned} I &= \frac{Crn}{100} \\ 80 &= \frac{250 \times 8 \times n}{100} \\ 8000 &= 2000n \\ n &= 4 \end{aligned}$$

It will take 4 years.

c) What rate per year must be paid for a capital of £750 to earn interest of £180 in four years?

$$\begin{aligned} I &= \frac{Crn}{100} \\ 180 &= \frac{750 \times r \times 4}{100} \\ 180 &= 30r \\ r &= 6\% \end{aligned}$$

A rate of 6% is required.

The total amount, A, after simple interest is added is given by the formula:

$$A = C + \frac{Crn}{100}$$

This is an example of an arithmetic sequence. These are covered in more detail in Topic 2.

Exercise 1.9

All rates of interest are annual rates.

1. Find the simple interest paid in each of the following cases:

Capital	Rate	Time period
a) NZ$300	6%	4 years
b) £750	8%	7 years
c) ¥425	6%	4 years
d) 2800 baht	4.5%	2 years
e) HK$880	6%	7 years

2. How long will it take for the following amounts of interest to be earned?

C	r	I
a) 500 baht	6%	150 baht
b) ¥5800	4%	¥ 96
c) AU$4000	7.5%	AU$1500
d) £2800	8.5%	£1904
e) €900	4.5%	€243
f) 400 Ft	9%	252 Ft

3. Calculate the rate of interest per year which will earn the given amount of interest in the stated time period:

Capital	Time period	Interest
a) €400	4 years	€1120
b) US$800	7 years	US$224
c) 2000 baht	3 years	210 baht
d) £1500	6 years	£675
e) €850	5 years	€340
f) AU$1250	2 years	AU$275

4. Calculate the capital that will earn the interest stated, in the number of years and at the given rate in each of the following cases:

Interest	Time period	Rate
a) 80 Ft	4 years	5%
b) NZ$36	3 years	6%
c) €340	5 years	8%
d) 540 baht	6 years	7.5%
e) €540	3 years	4.5%
f) US$348	4 years	7.25%

5. What rate of interest is paid on a deposit of £2000 that earns £400 interest in five years?

6. How long will it take a capital of €350 to earn €56 interest at 8% per year?

7. A capital of 480 Ft earns 108 Ft interest in five years. What rate of interest was being paid?
8. A capital of €750 becomes a total of €1320 in eight years. What rate of interest was being paid?
9. AU$1500 is invested for six years at 3.5% per year. What is the interest earned?
10. 500 baht is invested for 11 years and becomes 830 baht in total. What rate of interest was being paid?

■ Compound interest

Compound interest means interest is paid not only on the capital amount, but also on the interest itself: it is compounded (or added to).

This sounds complicated but the example below will make it clear.

e.g. A builder is going to build six houses on a plot of land in Spain. He borrows €500 000 at 10% interest and will pay off the loan in full after three years.

At the end of the first year he will owe:
€500 000 + 10% of €500 000 i.e. €500 000 × 1.10 = €550 000

At the end of the second year he will owe:
€550 000 + 10% of €550 000 i.e. €550 000 × 1.10 = €605 000

At the end of the third year he will owe:
€605 000 + 10% of €605 000 i.e. €605 000 × 1.10 = €665 500

The amount of interest he has to pay is €665 500 – €500 000 i.e. €165 500

The simple interest is €50 000 per year, i.e. a total of €150 000.

The difference of €15 500 is the compound interest.

The time taken for a debt to grow at compound interest can be calculated as shown in the example below:

Worked example How long will it take for a debt to double at a compound interest of 27% p.a.?

An interest rate of 27% implies a multiplier of 1.27.

Time (Years)	0	1	2	3
Debt	C	$1.27C$	$1.27^2C = 1.61C$	$1.27^3C = 2.05C$

The debt will have more than doubled after 3 years.

Using the example above of the builder's loan, if C represents capital he borrows, then after 1 year his debt will be given by the formula:

$D = C\left(1 + \frac{r}{100}\right)$ where r is the rate of interest

After 2 years: $D = C\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$

After 3 years: $D = C\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$

After n years: $D = C\left(1 + \frac{r}{100}\right)^n$

This formula for the debt includes the original capital loan. By subtracting C, the compound interest is calculated:

$$I = C\left(1 + \frac{r}{100}\right)^n - C$$

Compound interest is an example of a geometric sequence. Geometric sequences are covered in more detail in Topic 2.

The interest is usually calculated annually, but there can be other time periods. Compound interest can be charged yearly, half-yearly, quarterly, monthly or daily. (In theory any time period can be chosen.)

Worked examples

a) Use your graphics calculator to find the compound interest paid on a loan of \$600 for 3 years at an annual percentage rate (APR) of 5%.

The total payment is \$694.58 so the interest due is \$694 – \$600 = \$94.58.

b) Use your graphics calculator to find the compound interest when \$3000 is invested for 18 months at an APR of 8.5%. The interest is calculated every six months.

Note: The interest for each time period of 6 months is $\frac{8.5}{2}$%. There will therefore be 3 time periods of 6 months each.

The final sum is \$3399, so the interest is \$3399 – \$3000 = \$399.

Exercise 1.10

1. A shipping company borrows £70 million at 5% p.a. compound interest to build a new cruise ship. If it repays the debt after 3 years, how much interest will the company pay?
2. A woman borrows €100 000 for home improvements. The interest rate is 15% and she repays it in full after 3 years. How much interest will she pay?
3. A man owes \$5000 on his credit cards. The APR is 20%. If he doesn't repay any of the debt, how much will he owe after 4 years?
4. A school increases its intake by 10% each year. If it starts with 1000 students, how many will it have at the beginning of the fourth year of expansion?
5. 8 million tonnes of fish were caught in the North Sea in 2005. If the catch is reduced by 20% each year for 4 years, what weight is caught at the end of this time?
6. How many years will it take for a debt to double at 42% p.a. compound interest?
7. How many years will it take for a debt to double at 15% p.a. compound interest?
8. A car loses value at a rate of 27% each year. How long will it take for its value to halve?

Reverse percentages

Worked examples

a) In a test Ahmed answered 92% of the questions correctly. If he answered 23 questions correctly, how many did he get wrong?

> 92% of the marks is equivalent to 23 questions.
> 1% of the marks therefore is equivalent to $\frac{23}{92}$ questions.

So 100% is equivalent to $\frac{23}{92} \times 100 = 25$ questions.

Ahmed got 2 questions wrong.

b) A boat is sold for £15 360. This represents a profit of 28% to the seller. What did the boat originally cost the seller?

> The selling price is 128% of the original cost to the seller.
> 128% of the original cost is £15 360.
>
> 1% of the original cost is $\dfrac{£15\,360}{128}$.
>
> 100% of the original cost is $\dfrac{£15\,360}{128} \times 100$, i.e. £12 000.

Exercise 1.11

1. Calculate the value of X in each of the following:
   - a) 40% of X is 240
   - b) 24% of X is 84
   - c) 85% of X is 765
   - d) 4% of X is 10
   - e) 15% of X is 18.75
   - f) 7% of X is 0.105
2. Calculate the value of Y in each of the following:
   - a) 125% of Y is 70
   - b) 140% of Y is 91
   - c) 210% of Y is 189
   - d) 340% of Y is 68
   - e) 150% of Y is 0.375
   - f) 144% of Y is -54.72
3. In a Geography textbook, 35% of the pages are coloured. If there are 98 coloured pages, how many pages are there in the whole book?
4. A town has 3500 families who own a car. If this represents 28% of the families in the town, how many families are there in total?
5. In a test Isabel scored 88%. If she got three questions incorrect, how many did she get correct?
6. Water expands when it freezes. Ice is less dense than water so it floats. If the increase in volume is 4%, what volume of water will make an iceberg of $12\,700\,000\,\text{m}^3$? Give your answer to three significant figures.

SECTION
5

Approximation and rounding

In many instances exact numbers are not necessary or even desirable. In those circumstances approximations are given. The approximations can take several forms. The common types of approximations are outlined below.

Rounding

If 28 617 people attend a gymnastics competition, this figure can be reported to various levels of accuracy.

To the nearest 10 000 this figure would be rounded up to 30 000.
To the nearest 1000 the figure would be rounded up to 29 000.
To the nearest 100 the figure would be rounded down to 28 600.

In this type of situation it is unlikely that the exact number would be reported.

Exercise 1.12

1. Round the following numbers to the nearest 1000:
a) 68 786 b) 74 245 c) 89 000
d) 4020 e) 99 500 f) 999 999

2. Round the following numbers to the nearest 100:
a) 78 540 b) 6858 c) 14 099
d) 8084 e) 950 f) 2984

3. Round the following numbers to the nearest 10:
a) 485 b) 692 c) 8847
d) 83 e) 4 f) 997

Decimal places

A number can also be approximated to a given number of decimal places (d.p.). This refers to the number of digits written after a decimal point.

Worked examples

a) Write 7.864 to 1 d.p.

The answer needs to be written with one digit after the decimal point. However, to do this, the second digit after the decimal point also needs to be considered. If it is 5 or more then the first digit is rounded up.

i.e. 7.864 is written as 7.9 to 1 d.p.

b) Write 5.574 to 2 d.p.

The answer here is to be given with two digits after the decimal point. In this case the third digit after the decimal point needs to be considered. As the third digit after the decimal point is less than 5, the second digit is not rounded up.

i.e. 5.574 is written as 5.57 to 2 d.p.

Exercise 1.13

1. Round the following to 1 d.p.:
a) 5.58 b) 0.73 c) 11.86
d) 157.39 e) 4.04 f) 15.045
g) 2.95 h) 0.98 i) 12.049

2. Round the following to 2 d.p.:
a) 6.473 b) 9.587 c) 16.476
d) 0.088 e) 0.014 f) 9.3048
g) 99.996 h) 0.0048 i) 3.0037

Significant figures

Numbers can also be approximated to a given number of significant figures (s.f.). In the number 43.25 the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths.

Worked examples

a) Write 43.25 to 3 s.f.

Only the three most significant digits are written, however the fourth digit needs to be considered to see whether the third digit is to be rounded up or not.

i.e. 43.25 is written as 43.3 to 3 s.f.

b) Write 0.0043 to 1 s.f.

In this example only two digits have any significance, the 4 and the 3. The 4 is the most significant and therefore is the only one of the two to be written in the answer.

i.e. 0.0043 is written as 0.004 to 1 s.f.

Exercise 1.14

1. Write the following to the number of significant figures written in brackets:

a) 48 599 (1 s.f.) b) 48 599 (3 s.f.) c) 6841 (1 s.f.)
d) 7538 (2 s.f.) e) 483.7 (1 s.f.) f) 2.5728 (3 s.f.)
g) 990 (1 s.f.) h) 2045 (2 s.f.) i) 14.952 (3 s.f.)

2. Write the following to the number of significant figures written in brackets:

a) 0.085 62 (1 s.f.) b) 0.5932 (1 s.f.) c) 0.942 (2 s.f.)
d) 0.954 (1 s.f.) e) 0.954 (2 s.f.) f) 0.003 05 (1 s.f.)
g) 0.003 05 (2 s.f.) h) 0.009 73 (2 s.f.) i) 0.009 73 (1 s.f.)

Appropriate accuracy

In many instances calculations carried out using a calculator produce answers which are not whole numbers. A calculator will give the answer to as many decimal places as will fit on its screen. In most cases this degree of accuracy is neither desirable nor necessary. Unless another degree of accuracy is stated, answers involving lengths should be given to three significant figures and angles to one decimal place.

Worked example

Calculate 4.64 ÷ 2.3 giving your answer to an appropriate degree of accuracy.

The calculator will give the answer to 4.64 ÷ 2.3 as 2.0173913. However the answer given to 3 s.f. is sufficient. Therefore 4.64 ÷ 2.3 = 2.02 (3 s.f.).

Estimating answers to calculations

Even though many calculations can be done quickly and effectively on a calculator, often an estimate for an answer can be a useful check. This is done by rounding each of the numbers in such a way that the calculation becomes relatively straightforward.

Worked examples

a) Estimate the answer to 57×246.

Here are two possibilities:

i) $60 \times 200 = 12\,000$
ii) $50 \times 250 = 12\,500$.

b) Estimate the answer to $6386 \div 27$.

$6000 \div 30 = 200$

Exercise 1.15

1. Calculate the following, giving your answer to an appropriate degree of accuracy:
a) 23.456×17.89 b) 0.4×12.62 c) 18×9.24
d) $76.24 \div 3.2$ e) 7.6^2 f) 16.42^3
g) $\dfrac{2.3 \times 3.37}{4}$ h) $\dfrac{8.31}{2.02}$ i) $9.2 \div 4^2$

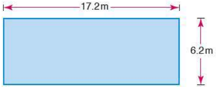
2. Without using a calculator, estimate the answers to the following:
a) $78.45 + 51.02$ b) $168.3 - 87.09$ c) 2.93×3.14

3. Without using a calculator, estimate the answers to the following:
a) 62×19 b) 270×12 c) 55×60
d) 4950×28 e) 0.8×0.95 f) 0.184×475

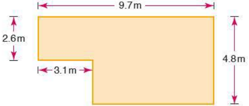
4. Without using a calculator, estimate the answers to the following:
a) $3946 \div 18$ b) $8287 \div 42$ c) $906 \div 27$
d) $5520 \div 13$ e) $48 \div 0.12$ f) $610 \div 0.22$

5. Estimate the shaded areas of the following shapes. Do *not* work out an exact answer.

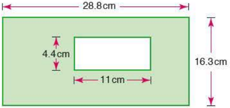
a)



b)

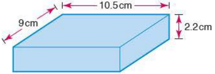


c)

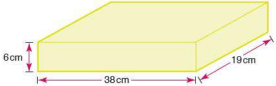


6. Estimate the volume of each of the solids below. Do *not* work out an exact answer.

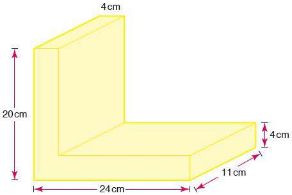
a)



b)



c)



SECTION 6

Standard form

In 1610 Galileo and a German astronomer, Marius, independently discovered Jupiter's four largest moons, Io, Europa, Ganymede and Callisto. At that time it was believed that the Sun revolved around the Earth. Galileo was one of the few people who believed that the Earth revolved around the Sun. As a result of this, the Church declared that he was a heretic and imprisoned him. It took the Church a further 350 years to officially accept that he was correct; he was only pardoned in 1992.

Facts about Jupiter:

It has a mass of 1 900 000 000 000 000 000 000 000 000 000 kg
It has a diameter of 142 800 000 m
It has a mean distance from the Sun of 778 000 000 km

Standard form is also known as standard index form or sometimes as scientific notation. It involves writing large numbers or very small numbers in terms of powers of 10.

A positive index

$$100 = 1 \times 10^2$$
$$1000 = 1 \times 10^3$$
$$10\,000 = 1 \times 10^4$$
$$3000 = 3 \times 10^3$$

For a number to be in standard form it must take the form $a \times 10^n$ where the index n is a positive or negative integer and a must lie in the range $1 \leq a < 10$.

e.g. 3100 can be written in many different ways:

3.1×10^3 $\quad 31 \times 10^2$ $\quad 0.31 \times 10^4$ etc.

However, only 3.1×10^3 agrees with the above conditions and therefore is the only one which is written in standard form.

Worked examples

a) Write 72 000 in standard form.

7.2×10^4

b) Of the numbers below, ring those which are written in standard form:

$\boxed{4.2 \times 10^3}$ $\quad 0.35 \times 10^2$ $\quad 18 \times 10^5$ $\quad \boxed{6 \times 10^3}$ $\quad 0.01 \times 10^1$

c) Multiply the following and write your answer in standard form:

600×4000
$= 2400000$
$= 2.4 \times 10^6$

d) Multiply the following and write your answer in standard form:

$(2.4 \times 10^4) \times (5 \times 10^7)$
$= 12 \times 10^{11}$
$= 1.2 \times 10^{12}$ when written in standard form

e) Divide the following and write your answer in standard form:

$(6.4 \times 10^7) \div (1.6 \times 10^3)$
$= 4 \times 10^4$

f) Add the following and write your answer in standard form:

$(3.8 \times 10^6) + (8.7 \times 10^4)$

Changing the indices to the same value gives the sum:

$(380 \times 10^4) + (8.7 \times 10^4)$
$= 388.7 \times 10^4$
$= 3.887 \times 10^6$ when written in standard form

g) Subtract the following and write your answer in standard form:

$(6.5 \times 10^7) - (9.2 \times 10^5)$

Changing the indices to the same value gives the sum:

$(650 \times 10^5) - (9.2 \times 10^5)$
$= 640.8 \times 10^5$
$= 6.408 \times 10^7$ when written in standard form

Your calculators have a standard form button and will also give answers in standard form if the answer is very large. For example, to enter the number 8×10^4 into the calculator, use the following keys on your calculator:

Casio	Texas
	
Note: A number such as 1 000 000 000 000 000 would appear on the screen as 1E + 15	Note: A number such as 1 000 000 000 000 000 would appear on the screen as 1 E 15

Exercise 1.16

1. Which of the following are not in standard form?

a) 6.2×10^5 b) 7.834×10^{16}
c) 8.0×10^5 d) 0.46×10^7
e) 82.3×10^6 f) 6.75×10^1

2. Write the following numbers in standard form:

a) 600 000 b) 48 000 000
c) 784 000 000 000 d) 534 000
e) 7 million f) 8.5 million

3. Write the following in standard form:

a) 68×10^5 b) 720×10^6
c) 8×10^5 d) 0.75×10^8
e) 0.4×10^{10} f) 50×10^6

4. Multiply the following and write your answers in standard form:

a) 200×3000 b) 6000×4000
c) 7 million × 20 d) 500 × 6 million
e) 3 million × 4 million f) 4500×4000

5. Light from the Sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^8 m/s, calculate to three significant figures (s.f.) the distance from the Sun to the Earth.

6. Find the value of the following and write your answers in standard form:

a) $(4.4 \times 10^3) \times (2 \times 10^5)$ b) $(6.8 \times 10^7) \times (3 \times 10^3)$
c) $(4 \times 10^5) \times (8.3 \times 10^5)$ d) $(5 \times 10^9) \times (8.4 \times 10^{12})$
e) $(8.5 \times 10^6) \times (6 \times 10^{15})$ f) $(5.0 \times 10^{12})^2$

7. Find the value of the following and write your answers in standard form:

a) $(3.8 \times 10^8) \div (1.9 \times 10^6)$ b) $(6.75 \times 10^9) \div (2.25 \times 10^4)$
c) $(9.6 \times 10^{11}) \div (2.4 \times 10^5)$ d) $(1.8 \times 10^{12}) \div (9.0 \times 10^7)$
e) $(2.3 \times 10^{11}) \div (9.2 \times 10^4)$ f) $(2.4 \times 10^8) \div (6.0 \times 10^3)$

8. Find the value of the following and write your answers in standard form:

a) $(3.8 \times 10^5) + (4.6 \times 10^4)$ b) $(7.9 \times 10^9) + (5.8 \times 10^8)$
c) $(6.3 \times 10^7) + (8.8 \times 10^5)$ d) $(3.15 \times 10^9) + (7.0 \times 10^6)$
e) $(5.3 \times 10^8) - (8.0 \times 10^7)$ f) $(6.5 \times 10^7) - (4.9 \times 10^6)$
g) $(8.93 \times 10^{10}) - (7.8 \times 10^9)$ h) $(4.07 \times 10^7) - (5.1 \times 10^6)$

9. The following list shows the distance of the planets of the Solar System from the Sun.

Jupiter	778 million km
Mercury	58 million km
Mars	228 million km
Uranus	2870 million km
Venus	108 million km
Neptune	4500 million km
Earth	150 million km
Saturn	1430 million km

Write each of the distances in standard form and then arrange them in order of magnitude, starting with the distance of the planet closest to the Sun.

A negative index

A negative index is used when writing a number between 0 and 1 in standard form.

e.g. $100 = 1 \times 10^{2}$
$10 = 1 \times 10^{1}$
$1 = 1 \times 10^{0}$
$0.1 = 1 \times 10^{-1}$
$0.01 = 1 \times 10^{-2}$
$0.001 = 1 \times 10^{-3}$
$0.0001 = 1 \times 10^{-4}$

Note that a must still lie within the range $1 \leq a < 10$.

Worked examples

a) Write 0.0032 in standard form.

3.2×10^{-3}

b) Write the following numbers in order of magnitude, starting with the largest:

3.6×10^{-3} 5.2×10^{-5} 1×10^{-2} 8.35×10^{-2} 6.08×10^{-8}

8.35×10^{-2}, 1×10^{-2}, 3.6×10^{-3}, 5.2×10^{-5}, 6.08×10^{-8}

Exercise 1.17

1. Write the following numbers in standard form:
a) 0.0006 b) 0.000053
c) 0.000864 d) 0.000000088
e) 0.0000007 f) 0.0004145

2. Write the following numbers in standard form:
a) 68×10^{-5} b) 750×10^{-9}
c) 42×10^{-11} d) 0.08×10^{-7}
e) 0.057×10^{-9} f) 0.4×10^{-10}

3. Deduce the value of n in each of the following cases:
a) $0.000\,25 = 2.5 \times 10^n$
b) $0.003\,57 = 3.57 \times 10^n$
c) $0.000\,000\,06 = 6 \times 10^n$
d) $0.004^2 = 1.6 \times 10^n$
e) $0.000\,65^2 = 4.225 \times 10^n$
f) $0.0002^n = 8 \times 10^{-12}$

4. Write these numbers in order of magnitude, starting with the largest:

3.2×10^{-4} 6.8×10^5 5.57×10^{-9} 6.2×10^3
5.8×10^{-7} 6.741×10^{-4} 8.414×10^2

SECTION 7

Speed, distance and time

Students need to be aware of the following formulae:

distance = speed × time

Rearranging the formula gives:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Where the speed is not constant:

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

Exercise 1.18

1. Find the average speed of an object moving:
a) 30 m in 5 s
b) 48 m in 12 s
c) 78 km in 2 h
d) 50 km in 2.5 h
e) 400 km in 2 h 30 min
f) 110 km in 2 h 12 min

2. How far will an object travel during:
a) 10 s at 40 m/s
b) 7 s at 26 m/s
c) 3 hours at 70 km/h
d) 4 h 15 min at 60 km/h
e) 10 min at 60 km/h
f) 1 h 6 min at 20 m/s?

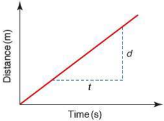
3. How long will it take to travel:
a) 50 m at 10 m/s
b) 1 km at 20 m/s
c) 2 km at 30 km/h
d) 5 km at 70 m/s
e) 200 cm at 0.4 m/s
f) 1 km at 15 km/h?

4. A train travels a distance of 420 km. The journey takes $3\frac{1}{2}$ hours and includes two stops each of 15 minutes. Calculate the average speed of the train:
a) for the whole journey
b) when it is moving.

5. A plane flies from Boston USA to London, a distance of 5600 km. It leaves at 8p.m. Boston local time and arrives at 8a.m. local time in London. If the time difference is 5 hours, calculate the average speed of the plane.

6. How long does it take a plane to fly from New Delhi to Sydney a distance of 10 420 km, if the plane flies at an average speed of 760 km/h. Give your answer:
a) to 2 decimal places
b) to the nearest minute.

7. A train leaves Paris at 8p.m. Monday and travels to Istanbul, a distance of 4200 km. If the train travels at an average speed of 70 km/h and the time difference is two hours, give the day and time at which the train arrives in Istanbul.

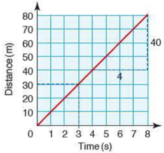


The graph of an object travelling at a constant speed is a straight line as shown.

$$\text{Gradient} = \frac{d}{t}$$

The units of the gradient are m/s, hence the gradient of a distance–time graph represents the speed at which the object is travelling.

Worked example



The graph shown represents an object travelling at constant speed.

i) From the graph, calculate how long it took to cover a distance of 30 m.
The time taken to travel 30 m is 3 seconds.

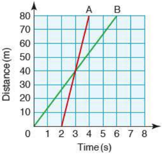
ii) Calculate the gradient of the graph.
Taking two points on the line, gradient = $\frac{40}{4} = 10$.

iii) Calculate the speed at which the object was travelling.
Gradient of a distance–time graph = speed.
Therefore the speed is 10 m/s.

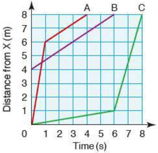
Exercise 1.19

1. Draw a distance–time graph for the first 10 seconds of an object travelling at 6 m/s.

2. Draw a distance–time graph for the first 10 seconds of an object travelling at 5 m/s. Use your graph to estimate:
   a) the time taken to travel 25 m,
   b) how far the object travels in 3.5 seconds.



3. Two objects A and B set off from the same point and move in the same straight line. B sets off first, whilst A sets off 2 seconds later.
   Use the distance–time graph shown to estimate:
   a) the speed of each of the objects
   b) how far apart the objects would be 20 seconds after the start.



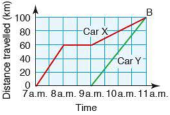
4. Three objects A, B and C move in the same straight line away from a point X.
   Both A and C change their speed during the journey, whilst B travels at the same constant speed throughout. From the distance–time graph estimate:
   a) the speed of object B
   b) the two speeds of object A
   c) the average speed of object C
   d) how far object C is from X 3 seconds from the start
   e) how far apart objects A and C are 4 seconds from the start.

Travel graphs

The graphs of two or more journeys can be shown on the same axes. The shape of the graph gives a clear picture of the movement of each of the objects.

Worked example

Car X and Car Y both reach point B 100 km from A at 11 a.m.



i) Calculate the speed of Car X between 7 a.m. and 8 a.m.

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{60}{1}\text{ km/h}$$

$$= 60\text{ km/h}$$

ii) Calculate the speed of Car Y between 9a.m. and 11a.m.

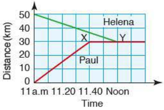
$$\text{speed} = \frac{100}{2}\,\text{km/h}$$
$$= 50\,\text{km/h}$$

iii) Explain what is happening to Car X between 8a.m. and 9 a.m.

No distance has been travelled, therefore Car X is stationary.

Exercise 1.20

1. Two friends Paul and Helena arrange to meet for lunch at noon. They live 50 km apart and the restaurant is 30 km from Paul's home. The travel graph below illustrates their journeys.



a) What is Paul's average speed between 11a.m. and 11.40a.m.?

b) What is Helena's average speed between 11a.m. and noon?

c) What does the line XY represent?

2. A car travels at a speed of 60 km/h for 1 hour. It then stops for 30 minutes and then continues at a constant speed of 80 km/h for a further 1.5 hours. Draw a distance–time graph for this journey.

3. A girl cycles for 1.5 hours at 10 km/h. She then stops for an hour and then travels for a further 15 km in 1 hour. Draw a distance–time graph of the girl's journey.

4. Two friends leave their houses at 4p.m. The houses are 4 km apart and the friends travel towards each other on the same road. Fyodor walks at 7 km/h and Yin at 5 km/h.

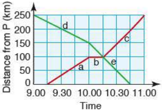
a) On the same axes, draw a distance–time graph of their journeys.

b) From your graph estimate the time at which they meet.

c) Estimate the distance from Fyodor's house to the point where they meet.

5. A train leaves a station P at 6p.m. and travels to station Q 150 km away. It travels at a steady speed of 75 km/h. At 6.10p.m. another train leaves Q for P at a steady speed of 100 km/h.
   a) On the same axes draw a distance–time graph to show both journeys.
   b) From the graph estimate the time at which both trains pass each other.
   c) At what distance from station Q do both trains pass each other?
   d) Which train arrives at its destination first?

6. A train sets off from town P at 9.15a.m. and heads towards town Q 250 km away. Its journey is split into the three stages a, b and c. At 9.00a.m. a second train left town Q heading for town P. Its journey was split into the two stages d and e. Using the graph below calculate the following:
   a) the speed of the first train during stages a, b and c,
   b) the speed of the second train during stages d and e.



SECTION 8

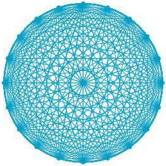
Investigations, modelling and ICT

The syllabus covered by this book examines investigations and modelling in Paper 6.

It can seem difficult to know how to begin an investigation. The suggestions below may help.

1. Read the question carefully and start with simple cases.
2. Draw simple diagrams to help.
3. Put the results from simple cases in a table.
4. Look for a pattern in your results.
5. Try to find a general rule in words.
6. Express your rule algebraically.
7. Test the rule for a new example.
8. Check that the original question has been answered.

Worked example



A mystic rose is created by placing a number of points evenly spaced on the circumference of a circle. Straight lines are then drawn from each point to every other point. The diagram (left) shows a mystic rose with 20 points.

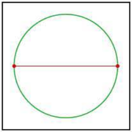
i) How many straight lines are there?
ii) How many straight lines would there be on a mystic rose with 100 points?

To answer these questions, you are not expected to draw either of the shapes and count the number of lines.

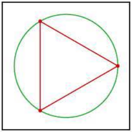
1/2. Try simple cases:

By drawing some simple cases and counting the lines, some results can be found:

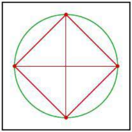
Mystic rose with 2 points
Number of lines = 1



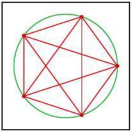
Mystic rose with 3 points
Number of lines = 3



Mystic rose with 4 points
Number of lines = 6



Mystic rose with 5 points
Number of lines = 10



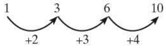
3. Enter the results in an ordered table:

Number of points.	2	3	4	5
Number of lines	1	3	6	10

4/5. Look for a pattern in the results:

There are two patterns.

The first shows how the values change.



It can be seen that the difference between successive terms is increasing by one each time.

The problem with this pattern is that to find the 20th and 100th terms, it would be necessary to continue this pattern and find all the terms leading up to the 20th and 100th term.

The second is the relationship between the number of points and the number of lines.

Number of points.	2	3	4	5
Number of lines	1	3	6	10

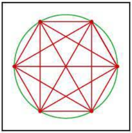
It is important to find a relationship that works for all values, for example subtracting 1 from the number of points gives the number of lines in the first example only, so is not useful. However, halving the number of points and multiplying this by 1 less than the number of points works each time,

i.e. Number of lines = (half the number of points) × (one less than the number of points).

6. **Express the rule algebraically:**
The rule expressed in words above can be written more elegantly using algebra. Let the number of lines be l and the number of points be p.

$$l = \tfrac{1}{2}p(p - 1)$$

Note: Any letters can be used to represent the number of lines and the number of points, not just l and p.



7. **Test the rule:**
The rule was derived from the original results. It can be tested by generating a further result.

If the number of points $p = 6$, then the number of lines l is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 6(6 - 1) \\ &= 3 \times 5 \\ &= 15 \end{aligned}$$

From the diagram to the left, the number of lines can also be counted as 15.

8. **Check that the original questions have been answered:**
Using the formula, the number of lines in a mystic rose with 20 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 20(20 - 1) \\ &= 10 \times 19 \\ &= 190 \end{aligned}$$

The number of lines in a mystic rose with 100 points is:

$$\begin{aligned} l &= \tfrac{1}{2} \times 100(100 - 1) \\ &= 50 \times 99 \\ &= 4950 \end{aligned}$$

Primes and squares

13, 41 and 73 are prime numbers.

Two different square numbers can be added together to make these prime numbers, e.g. $3^2 + 8^2 = 73$.

1. Find the two square numbers that can be added to make 13 and 41.
2. List the prime numbers less than 100.
3. Which of the prime numbers less than 100 can be shown to be the sum of two different square numbers?
4. Is there a rule to the numbers in Q.3?
5. Your rule is a predictive rule not a formula. Discuss the difference.

Spanish football league

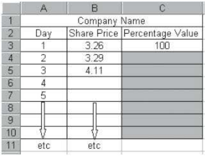
There are 18 teams in Series A of the Spanish football league.

1. If each team plays the other teams twice, once at home and once away, then how many matches are played in a season?
2. If there are *t* teams in a league, how many matches are played in a season?

ICT Activity 1

In this activity you will be using a spreadsheet to track the price of a company's shares over a period of time.

1. a) Using the internet or a newspaper as a resource, find the value of a particular company's shares.
   b) Over a period of a month (or week), record the value of the company's shares. This should be carried out on a daily basis.
2. When you have collected all the results, enter them into a spreadsheet similar to the one shown below:



	A	B	C
1	Company Name		
2	Day	Share Price	Percentage Value
3	1	3.26	100
4	2	3.29	
5	3	4.11	
6	4		
7	5		
8			
9			
10			
11	etc	etc	

3. In column C enter formulae that will calculate the value of the shares as a percentage of their value on day 1.

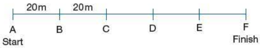
4. When the spreadsheet is complete, produce a graph showing how the percentage value of the share price changed over time.
5. Write a short report explaining the performance of the company's shares during that time.

ICT Activity 2

The following activity refers to the graphing package Autograph; however, a similar package may be used.

The velocity of a student at different parts of a 100 m sprint will be analysed.

A racecourse is set out as shown below:



1. A student must stand at each of points A–F. The student at A runs the 100 m and is timed as he/she runs past each of the points B–F by the students at these points who each have a stopwatch.
2. In Autograph, plot a distance–time graph of the results by entering the data as pairs of coordinates, i.e. (time, distance).
3. Ensure that all the points are selected and draw a curve of best fit through them.
4. Select the curve and plot a coordinate of your choice on it. This point can now be moved along the curve using the cursor keys on the keyboard.
5. Draw a tangent to the curve through the point.
6. What does the gradient of the tangent represent?
7. At what point of the race was the student running fastest? How did you reach this answer?
8. Collect similar data for other students. Compare their graphs and running speeds.
9. Carefully analyse one of the graphs and write a brief report to the runner in which you should identify, giving reasons, the parts of the race he/she needs to improve on.

SECTION 9 Student assessments

Student assessment 1

1. Explain, giving examples, the differences and similarities between a real number and a rational number.

2. Simplify the following expressions:

a) $3 \times \quad 6$ b) $5 \times \quad 7$ c) $2 \times \quad 12$

3. Simplify the following expressions:
   a) $\sqrt{3} + \sqrt{27}$ b) $\sqrt{24} + \sqrt{54}$ c) $3\sqrt{8} - \sqrt{32}$

4. Expand the following expressions and simplify as far as possible:
   a) $(1 - \sqrt{2})(3 + \sqrt{2})$ b) $(3\sqrt{5} - 2)^2$

5. Rationalise the following fractions:
   a) $\dfrac{3}{\sqrt{5}}$ b) $\dfrac{5}{\sqrt{10}}$ c) $\dfrac{4}{\sqrt{2} - 1}$
   d) $\dfrac{2 - \sqrt{3}}{2 + \sqrt{3}}$

6. a) A rod has a length of $\sqrt{3}$ cm. If 3 of these rods are placed end to end, decide whether the total length is a rational or irrational number. Give reasons for your answer.
   b) A square has side length $3\sqrt{5}$ cm. Decide whether the area of the square is rational or irrational. Give reasons for your answer.

Student assessment 2

1. Copy the table below and fill in the missing values:

Fraction	Decimal	Percentage
	0.25	
$\frac{3}{5}$		
		$62\frac{1}{2}\%$
$2\frac{1}{4}$		

2. Find 30% of 2500 m.

3. In a sale a shop reduces its prices by 12.5%. What is the sale price of a desk previously costing €600?

4. In the last six years the value of a house has increased by 35%. If it cost £72 000 six years ago, what is its value now?

5. Express the first quantity as a percentage of the second.
   a) 35 mins, 2 hours b) 650 g, 3 kg
   c) 5 m, 4 m d) 15 s, 3 mins
   e) 600 kg, 3 tonnes f) 35 cl, 3.5 l

6. Shares in a company are bought for $600. After a year the same shares are sold for $550. Calculate the percentage depreciation.

7. In a sale the price of a jacket originally costing 17 000 Japanese yen (¥) is reduced by ¥4000. Any item not sold by the last day of the sale is reduced by a further 50%. If the jacket is sold on the last day of the sale, calculate:
   a) the price it is finally sold for
   b) the overall percentage reduction in price.

8. The population of a type of insect increases by approximately 10% each day. How many days will it take for the population to double?

9. Find the compound interest on €5 million for 3 years at 6% interest p.a.

10. A boat loses 15% of its value each year. How long will it take for its value to halve?

Student assessment 3

1. Calculate the original price in each of the following:

Selling price	Profit
$224	12%
$62.50	150%
$660.24	26%
$38.50	285%

2. Calculate the original price in each of the following:

Selling price	Loss
$392.70	15%
$2480	38%
$3937.50	12.5%
$4675	15%

3. In an examination Sarah obtained 87.5% by gaining 105 marks. How many marks did she lose?

4. At the end of a year a factory has produced 38 500 television sets. If this represents a 10% increase in productivity on last year, calculate the number of sets that were made last year.

5. A computer manufacturer is expected to have produced 24 000 units by the end of this year. If this represents a 4% decrease on last year's output, calculate the number of units produced last year.

6. A farmer increased his yield by 5% each year over the last five years. If he produced 600 tonnes this year, calculate to the nearest tonne his yield five years ago.

Student assessment 4

1. Round the following numbers to the degree of accuracy shown in brackets:

   a) 6472 (nearest 10) b) 88 465 (nearest 100)
   c) 64 785 (nearest 1000) d) 6.7 (nearest 10)

2. Round the following numbers to the number of decimal places shown in brackets:

   a) 6.78 (1 d.p.) b) 4.438 (2 d.p.)
   c) 7.975 (1 d.p.) d) 63.084 (2 d.p.)
   e) 0.0567 (3 d.p.) f) 3.95 (2 d.p.)

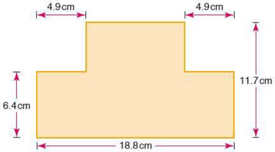
3. Round the following numbers to the number of significant figures shown in brackets:

   a) 42.6 (1 s.f.) b) 5.432 (2 s.f.)
   c) 0.0574 (1 s.f.) d) 48 572 (2 s.f.)
   e) 687 453 (1 s.f.) f) 687 453 (3 s.f.)

4. 1 mile is 1760 yards. Estimate the number of yards in 19 miles.

5. Estimate the area of the figure below:

NB: The diagram is not drawn to scale.



6. Estimate the answers to the following. Do *not* work out an exact answer.

   a) $\dfrac{3.9 \times 26.4}{4.85}$ b) $\dfrac{(3.2)^3}{(5.4)^2}$ c) $\dfrac{2.8 \times (7.3)^2}{(3.2)^2 \times 6.2}$

7. A cuboid's dimensions are given as 3.973 m by 2.4 m by 3.16 m. Calculate its volume, giving your answer to an appropriate degree of accuracy.

8. A girl runs a race in 14.2 seconds. If she rounds her time down to 14 seconds, what is her error as a percentage of her actual time?

9. a) Use a calculator to find the exact answer to Q.5.
b) Calculate your error as a percentage of the real area.

10. Show that the following numbers are rational:
a) 0.875 b) $\sqrt[3]{125}$ c) 0.44

Student assessment 5

1. Write the following numbers in standard form:
a) 6 million b) 0.0045
c) 3 800 000 000 d) 0.000 000 361
e) 460 million f) 3

2. Write the following numbers in order of magnitude, starting with the largest:
3.6×10^2 2.1×10^{-3} 9×10^1
4.05×10^8 1.5×10^{-2} 7.2×10^{-3}

3. Write the following numbers:
a) in standard form
b) in order of magnitude, starting with the smallest.
15 million 430 000 0.000 435 4.8 0.0085

4. Deduce the value of n in each of the following:
a) $4750 = 4.75 \times 10^n$ b) $6\,440\,000\,000 = 6.44 \times 10^n$
c) $0.0040 = 4.0 \times 10^n$ d) $1000^2 = 1 \times 10^n$
e) $0.9^3 = 7.29 \times 10^n$ f) $800^3 = 5.12 \times 10^n$

5. Write the answers to the following calculations in standard form:
a) $50\,000 \times 2400$ b) $(3.7 \times 10^6) \times (4.0 \times 10^4)$
c) $(5.8 \times 10^7) + (9.3 \times 10^6)$ d) $(4.7 \times 10^6) - (8.2 \times 10^5)$

6. The speed of light is 3×10^8 m/s. Jupiter is 778 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Jupiter.

7. A star is 300 light years away from Earth. If the speed of light is 3×10^5 km/s, calculate the distance from the star to Earth. Give your answer in kilometres and written in standard form.

Student assessment 6

1. A woman climbs to the top of a hill at an average vertical speed of 1.5 km/h. If she climbs for 4 hours and 40 minutes, with two half-hour breaks, calculate the height of the hill.

2. A cyclist completes a journey of 240 km in 8 hours.
a) Calculate his average speed.
b) If his average speed was 25% faster, how long would the journey have taken him?

3. A car travels at 60 km/h for 1 hour. The driver then takes a 30 minute break. After her break, she continues at 80 km/h for 90 minutes.

a) Draw a distance–time graph for her journey.

b) Calculate the total distance travelled.

4. Two trains depart at the same time from cities M and N, which are 200 km apart. One train travels from M to N, the other travels from N to M. The train departing from M travels a distance of 60 km in the first hour, 120 km in the next 1.5 hours and then the rest of the journey at 40 km/h. The train departing from N travels the whole distance at a speed of 100 km/h. Assuming all speeds are constant:

a) draw a travel graph to show both journeys

b) estimate how far from city M the trains are when they pass each other

c) estimate how long after the start of the journey it is when the trains pass each other.

5. A boy lives 3.5 km from his school. He walks home at a constant speed of 9 km/h for the first 10 minutes. He then stops and talks to his friends for 5 minutes. He finally runs the rest of his journey home at a constant speed of 12 km/h.

a) Illustrate this information on a distance–time graph.

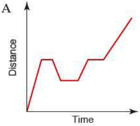
b) Use your graph to estimate the total time it took the boy to get home that day.

6. Below are four distance–time graphs A, B, C and D. Two of them are not possible.

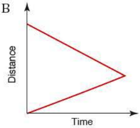
a) Which two graphs are impossible?

b) Explain why the two you have chosen are not possible.

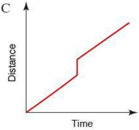
A



B



C



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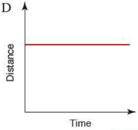


TOPIC

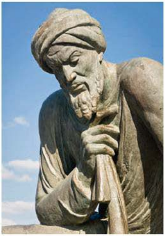
2 Algebra

This topic will cover the following syllabus content:

2.2 Solution of linear inequalities
Solution of inequalities using a graphics calculator
2.3 Solution of linear equations including those with fractional expressions
2.4 Indices
2.5 Derivation, rearrangement and evaluation of formulae
2.6 Solution of simultaneous linear equations in two variables
2.7 Expansion of brackets, including the square of a binomial
2.8 Factorisation: common factor; difference of squares; trinomial; four term
2.9 Algebraic fractions: simplification, including use of factorisation; addition or subtraction of fractions with linear denominators; multiplication or division and simplification of two fractions
2.10 Solution of quadratic equations: by factorisation; using a graphics calculator; using the quadratic formula
2.11 Use of a graphics calculator to solve equations (for unfamiliar equations see Topic 3)
2.12 Continuation of a sequence of numbers or patterns
Determination of the *n*th term
Use of a difference method to find the formula for a linear sequence, a quadratic sequence or a cubic sequence
Identification of a simple geometric sequence and determination of its formula
2.13 Direct variation $y \alpha x$, $y \alpha x^2$, $y \alpha x^3$, $y \alpha \sqrt{x}$
Inverse variation $y \alpha \frac{1}{x}$, $y \alpha \frac{1}{x^2}$, $y \alpha \frac{1}{\sqrt{x}}$
Best variation model for given data

Sections

SECTION 1 The Persians



Al-Khwarizmi (790–850)

Abu Ja'far Muhammad Ibn Musa Al-Khwarizmi is called the 'father of algebra'. He was born in Baghdad in AD790. He wrote the book *Hisab al-jabr w'al-muqabala* in AD830 when Baghdad had the greatest university in the world and the greatest mathematicians studied there. He gave us the word 'algebra' and worked on quadratic equations. He also introduced the decimal system from India.

Muhammad al-Karaji was born in North Africa in what is now Morocco. He lived in the eleventh century and worked on the theory of indices. He also worked on an algebraic method of calculating square and cube roots. He may also have travelled to the University in Granada (then part of the Moorish Empire) where works of his can be found in the University library.

The poet Omar Khayyam is known for his long poem *The Rubaiyat*. He was also a fine mathematician working on the binomial theorem. He introduced the symbol 'shay', which became our 'x'.

SECTION 2 Algebraic representation and manipulation

Expanding brackets

When removing brackets, every term inside the bracket must be multiplied by whatever is outside the bracket.

Worked examples

a) $3(x+4)$
$= 3x + 12$

b) $5x(2y+3)$
$= 10xy + 15x$

c) $2a(3a+2b-3c)$
$= 6a^2 + 4ab - 6ac$

d) $-4p(2p-q+r^2)$
$= -8p^2 + 4pq - 4pr^2$

e) $-2x^2\left(x+3y-\frac{1}{x}\right)$
$= -2x^3 - 6x^2y + 2x$

f) $\frac{-2}{x}\left(-x+4y+\frac{1}{x}\right)$
$= 2 - \frac{8y}{x} - \frac{2}{x^2}$

Exercise 2.1

Expand the following:

1. a) $4(x-3)$ b) $5(2p-4)$ c) $-6(7x-4y)$
d) $3(2a-3b-4c)$ e) $-7(2m-3n)$ f) $-2(8x-3y)$

2. a) $3x(x-3y)$ b) $a(a+b+c)$ c) $4m(2m-n)$
d) $-5a(3a-4b)$ e) $-4x(-x+y)$ f) $-8p(-3p+q)$

3. a) $-(2x^2-3y^2)$ b) $-(-a+b)$ c) $-(-7p+2q)$
d) $\frac{1}{2}(6x-8y+4z)$ e) $\frac{3}{4}(4x-2y)$ f) $\frac{1}{5}x(10x-15y)$

4. a) $3r(4r^2-5s+2t)$ b) $a^2(a+b+c)$ c) $3a^2(2a-3b)$
d) $pq(p+q-pq)$ e) $m^2(m-n+nm)$ f) $a^3(a^3+a^2b)$

Exercise 2.2

Expand and simplify the following:

1. a) $3a-2(2a+4)$ b) $8x-4(x+5)$
c) $3(p-4)-4$ d) $7(3m-2n)+8n$
e) $6x-3(2x-1)$ f) $5p-3p(p+2)$

2. a) $7m(m+4)+m^2+2$ b) $3(x-4)+2(4-x)$
c) $6(p+3)-4(p-1)$ d) $5(m-8)-4(m-7)$
e) $3a(a+2)-2(a^2-1)$ f) $7a(b-2c)-c(2a-3)$

3. a) $\frac{1}{2}(6x+4)+\frac{1}{3}(3x+6)$
b) $\frac{1}{4}(2x+6y)+\frac{3}{4}(6x-4y)$
c) $\frac{1}{8}(6x-12y)+\frac{1}{2}(3x-2y)$
d) $\frac{1}{5}(15x+10y)+\frac{3}{10}(5x-5y)$
e) $\frac{2}{3}(6x-9y)+\frac{1}{3}(9x+6y)$
f) $\frac{x}{7}(14x-21y)-\frac{x}{2}(4x-6y)$

Simple factorising

When factorising, the largest possible factor is removed from each of the terms and placed outside the brackets.

Worked examples Factorise the following expressions:

a) $10x+15$
$=5(2x+3)$

b) $8p-6q+10r$
$=2(4p-3q+5r)$

c) $-2q-6p+12$
$=2(-q-3p+6)$

d) $2a^2+3ab-5ac$
$=a(2a+3b-5c)$

e) $6ax-12ay-18a^2$
$=6a(x-2y-3a)$

f) $3b+9ba-6bd$
$=3b(1+3a-2d)$

Exercise 2.3

Factorise the following:

1. a) $4x - 6$ b) $18 - 12p$
c) $6y - 3$ d) $4a + 6b$
e) $3p - 3q$ f) $8m + 12n + 16r$

2. a) $3ab + 4ac - 5ad$ b) $8pq + 6pr - 4ps$
c) $a^2 - ab$ d) $4x^2 - 6xy$
e) $abc + abd + fab$ f) $3m^2 + 9m$

3. a) $3pqr - 9pqs$ b) $5m^2 - 10mn$
c) $8x^2y - 4xy^2$ d) $2a^2b^2 - 3b^2c^2$
e) $12p - 36$ f) $42x - 54$

4. a) $18 + 12y$ b) $14a - 21b$
c) $11x + 11xy$ d) $4s - 16t + 20r$
e) $5pq - 10qr + 15qs$ f) $4xy + 8y^2$

5. a) $m^2 + mn$ b) $3p^2 - 6pq$
c) $pqr + qrs$ d) $ab + a^2b + ab^2$
e) $3p^3 - 4p^4$ f) $7b^3c + b^2c^2$

6. a) $m^3 - m^2n + mn^2$ b) $4r^3 - 6r^2 + 8r^2s$
c) $56x^2y - 28xy^2$ d) $72m^2n + 36mn^2 - 18m^2n^2$

Substitution

Worked examples Evaluate the following expressions if $a = 3, b = 4, c = -5$:

a) $2a + 3b - c$
$= 6 + 12 + 5$
$= 23$

b) $3a - 4b + 2c$
$= 9 - 16 - 10$
$= -17$

c) $-2a + 2b - 3c$
$= -6 + 8 + 15$
$= 17$

d) $a^2 + b^2 + c^2$
$= 9 + 16 + 25$
$= 50$

e) $3a(2b - 3c)$
$= 9(8 + 15)$
$= 9 \times 23$
$= 207$

f) $-2c(-a + 2b)$
$= 10(-3 + 8)$
$= 10 \times 5$
$= 50$

Graphics calculators have a large number of memory channels. These can be used to store numbers which can then be substituted into an expression.

Using $a = 3, b = 4, c = -5$ as above, use your graphics calculator to evaluate $2a - 3b + c$.

Casio

to store 3 in memory channel A.

to store 4 and –5 in memory channels B and C respectively.

```
3→A
                3
4→B
                4
-5→C
               -5
▶MAT
```

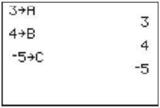
to evaluate 2*a* – 3*b* + *c*.

```
2A-3B+C
              -11
```

Texas

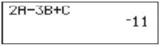
to store 3 in memory channel A.

to store 4 and –5 in memory channels B and C respectively.



```
3→A
                3
4→B
                4
-5→C
               -5
```

to evaluate 2*a* – 3*b* + *c*.



```
2A-3B+C
              -11
```

Exercise 2.4

Evaluate the following expressions if $p = 4$, $q = -2$, $r = 3$ and $s = -5$:

1. a) $2p + 4q$ b) $5r - 3s$
c) $3q - 4s$ d) $6p - 8q + 4s$
e) $3r - 3p + 5q$ f) $-p - q + r + s$

2. a) $2p - 3q - 4r + s$ b) $3s - 4p + r + q$
c) $p^2 + q^2$ d) $r^2 - s^2$
e) $p(q - r + s)$ f) $r(2p - 3q)$

3. a) $2s(3p - 2q)$ b) $pq + rs$
c) $2pr - 3rq$ d) $q^3 - r^2$
e) $s^3 - p^3$ f) $r^4 - q^5$

4. a) $-2pqr$ b) $-2p(q+r)$
c) $-2rq+r$ d) $(p+q)(r-s)$
e) $(p+s)(r-q)$ f) $(r+q)(p-s)$

5. a) $(2p+3q)(p-q)$ b) $(q+r)(q-r)$
c) q^2-r^2 d) p^2-r^2
e) $(p+r)(p-r)$ f) $(-s+p)q^2$

Rearranging formulae

In the formula $a = 2b + c$, 'a' is the subject. In order to make either b or c the subject, the formula has to be rearranged.

Worked examples Rearrange the following formulae to make the **red** letter the subject:

a) $a = 2b + \boldsymbol{c}$
$a - 2b = \boldsymbol{c}$

b) $2r + \boldsymbol{p} = q$
$\boldsymbol{p} = q - 2r$

c) $ab = \boldsymbol{c}d$
$\frac{ab}{d} = \boldsymbol{c}$

d) $\frac{a}{b} = \frac{c}{\boldsymbol{d}}$
$a\boldsymbol{d} = cb$
$\boldsymbol{d} = \frac{cb}{a}$

Exercise 2.5

In the following questions, make the letter in **red** the subject of the formula:

1. a) $m + \boldsymbol{n} = r$ b) $\boldsymbol{m} + n = p$
c) $2m + \boldsymbol{n} = 3p$ d) $3x = 2p + \boldsymbol{q}$
e) $\boldsymbol{a}b = cd$ f) $ab = c\boldsymbol{d}$

2. a) $3\boldsymbol{x}y = 4m$ b) $7pq = 5\boldsymbol{r}$
c) $3\boldsymbol{x} = c$ d) $3\boldsymbol{x} + 7 = y$
e) $5\boldsymbol{y} - 9 = 3r$ f) $5y - 9 = 3\boldsymbol{x}$

3. a) $6\boldsymbol{b} = 2a - 5$ b) $6b = 2\boldsymbol{a} - 5$
c) $3x - 7y = 4\boldsymbol{z}$ d) $3\boldsymbol{x} - 7y = 4z$
e) $3x - 7\boldsymbol{y} = 4z$ f) $2\boldsymbol{p}r - q = 8$

4. a) $\frac{\boldsymbol{p}}{4} = r$ b) $\frac{4}{\boldsymbol{p}} = 3r$
c) $\frac{1}{5}n = 2\boldsymbol{p}$ d) $\frac{1}{5}\boldsymbol{n} = 2p$
e) $\boldsymbol{p}(q + r) = 2t$ f) $p(\boldsymbol{q} + r) = 2t$

5. a) $3m - n = \boldsymbol{r}t(p + q)$ b) $3m - n = r\boldsymbol{t}(p + q)$
c) $3\boldsymbol{m} - n = rt(p + q)$ d) $3m - \boldsymbol{n} = rt(p + q)$
e) $3m - n = rt(\boldsymbol{p} + q)$ f) $3m - n = rt(p + \boldsymbol{q})$

6. a) $\frac{ab}{c} = \boldsymbol{d}e$ b) $\frac{\boldsymbol{a}b}{c} = de$
c) $\frac{ab}{\boldsymbol{c}} = de$ d) $\frac{\boldsymbol{a} + b}{c} = d$
e) $\frac{a}{c} + \boldsymbol{b} = d$ f) $\frac{a}{\boldsymbol{c}} + b = d$

SECTION 3 Further algebraic representation and manipulation

Further expansion

When multiplying together expressions in brackets, it is necessary to multiply all the terms in one bracket by all the terms in the other bracket.

Worked examples Expand the following:

a) $(x + 3)(x + 5)$

	x	$+3$
x	x^2	$3x$
$+5$	$5x$	15

$= x^2 + 5x + 3x + 15$
$= x^2 + 8x + 15$

b) $(x + 2)(x + 1)$

	x	$+1$
x	x^2	x
$+2$	$2x$	2

$= x^2 + x + 2x + 2$
$= x^2 + 3x + 2$

Exercise 2.6

Expand and simplify the following:

1. a) $(x + 2)(x + 3)$ b) $(x + 3)(x + 4)$
c) $(x + 5)(x + 2)$ d) $(x + 6)(x + 1)$
e) $(x - 2)(x + 3)$ f) $(x + 8)(x - 3)$

2. a) $(x - 4)(x + 6)$ b) $(x - 7)(x + 4)$
c) $(x + 5)(x - 7)$ d) $(x + 3)(x - 5)$
e) $(x + 1)(x - 3)$ f) $(x - 7)(x + 9)$

3. a) $(x - 2)(x - 3)$ b) $(x - 5)(x - 2)$
c) $(x - 4)(x - 8)$ d) $(x + 3)(x + 3)$
e) $(x - 3)(x - 3)$ f) $(x - 7)(x - 5)$

4. a) $(x + 3)(x - 3)$ b) $(x + 7)(x - 7)$
c) $(x - 8)(x + 8)$ d) $(x + y)(x - y)$
e) $(a + b)(a - b)$ f) $(p - q)(p + q)$

5. a) $(y + 2)(2y + 3)$ b) $(y + 7)(3y + 4)$
c) $(2y + 1)(y + 8)$ d) $(2y + 1)(2y + 2)$
e) $(3y + 4)(2y + 5)$ f) $(6y + 3)(3y + 1)$

6. a) $(2p-3)(p+8)$
b) $(4p-5)(p+7)$
c) $(3p-4)(2p+3)$
d) $(4p-5)(3p+7)$
e) $(6p+2)(3p-1)$
f) $(7p-3)(4p+8)$

7. a) $(2x-1)(2x-1)$
b) $(3x+1)^2$
c) $(4x-2)^2$
d) $(5x-4)^2$
e) $(2x+6)^2$
f) $(2x+3)(2x-3)$

8. a) $(3+2x)(3-2x)$
b) $(4x-3)(4x+3)$
c) $(3+4x)(3-4x)$
d) $(7-5y)(7+5y)$
e) $(3+2y)(4y-6)$
f) $(7-5y)^2$

Further factorisation

Factorisation by grouping

Worked examples Factorise the following expressions:

a) $6x+3+2xy+y$

There is no common factor to all four terms, however pairs of terms can be factorised.

$= 3(2x+1)+y(2x+1)$
$= (3+y)(2x+1)$

Note that $(2x+1)$ was a common factor of both terms.

b) $ax+ay-bx-by$
$= a(x+y)-b(x+y)$
$= (a-b)(x+y)$

c) $2x^2-3x+2xy-3y$
$= x(2x-3)+y(2x-3)$
$= (x+y)(2x-3)$

Exercise 2.7

Factorise the following by grouping:

1. a) $ax+bx+ay+by$
b) $ax+bx-ay-by$
c) $3m+3n+mx+nx$
d) $4m+mx+4n+nx$
e) $3m+mx-3n-nx$
f) $6x+xy+6z+zy$

2. a) $pr-ps+qr-qs$
b) $pq-4p+3q-12$
c) $pq+3q-4p-12$
d) $rs+rt+2ts+2t^2$
e) $rs-2ts+rt-2t^2$
f) $ab-4cb+ac-4c^2$

3. a) $xy+4y+x^2+4x$
b) $x^2-xy-2x+2y$
c) $ab+3a-7b-21$
d) $ab-b-a+1$
e) $pq-4p-4q+16$
f) $mn-5m-5n+25$

4. a) $mn-2m-3n+6$
b) $mn-2mr-3rn-6r^2$
c) $pr-4p-4qr+16q$
d) $ab-a-bc+c$
e) $x^2-2xz-2xy+4yz$
f) $2a^2+2ab+b^2+ab$

Difference of two squares

On expanding: $(x + y)(x - y)$

$$= x^2 - xy + xy - y^2$$
$$= x^2 - y^2$$

The reverse is that $x^2 - y^2$ factorises to $(x + y)(x - y)$. x^2 and y^2 are both square and therefore $x^2 - y^2$ is known as the **difference of two squares**.

Worked examples

a) $p^2 - q^2$
$= (p + q)(p - q)$

b) $4a^2 - 9b^2$
$= (2a)^2 - (3b)^2$
$= (2a + 3b)(2a - 3b)$

c) $(mn)^2 - 25k^2$
$= (mn)^2 - (5k)^2$
$= (mn + 5k)(mn - 5k)$

d) $4x^2 - (9y)^2$
$= (2x)^2 - (9y)^2$
$= (2x + 9y)(2x - 9y)$

Exercise 2.8 Factorise the following:

1. a) $a^2 - b^2$ b) $m^2 - n^2$ c) $x^2 - 25$
d) $m^2 - 49$ e) $81 - x^2$ f) $100 - y^2$

2. a) $144 - y^2$ b) $q^2 - 169$ c) $m^2 - 1$
d) $1 - t^2$ e) $4x^2 - y^2$ f) $25p^2 - 64q^2$

3. a) $9x^2 - 4y^2$ b) $16p^2 - 36q^2$ c) $64x^2 - y^2$
d) $x^2 - 100y^2$ e) $(pq)^2 - 4p^2$ f) $(ab)^2 - (cd)^2$

4. a) $m^2n^2 - 9y^2$ b) $\frac{1}{4}x^2 - \frac{1}{9}y^2$ c) $p^4 - q^4$
d) $4m^4 - 36y^4$ e) $16x^4 - 81y^4$ f) $(2x)^2 - (3y)^4$

Evaluation

Once factorised, numerical expressions can be evaluated.

Worked examples Evaluate the following expressions:

a) $13^2 - 7^2$
$= (13 + 7)(13 - 7)$
$= 20 \times 6$
$= 120$

b) $6.25^2 - 3.75^2$
$= (6.25 + 3.75)(6.25 - 3.75)$
$= 10 \times 2.5$
$= 25$

Exercise 2.9 By factorising, evaluate the following:

1. a) $8^2 - 2^2$ b) $16^2 - 4^2$ c) $49^2 - 1$
d) $17^2 - 3^2$ e) $88^2 - 12^2$ f) $96^2 - 4^2$

2. a) $45^2 - 25$ b) $99^2 - 1$ c) $27^2 - 23^2$
d) $66^2 - 34^2$ e) $999^2 - 1$ f) $225 - 8^2$

3. a) $8.4^2 - 1.6^2$ b) $9.3^2 - 0.7^2$ c) $42.8^2 - 7.2^2$
d) $(8\frac{1}{2})^2 - (1\frac{1}{2})^2$ e) $(7\frac{3}{4})^2 - (2\frac{1}{4})^2$ f) $5.25^2 - 4.75^2$

4. a) $8.62^2 - 1.38^2$ b) $0.9^2 - 0.1^2$ c) $3^4 - 2^4$
d) $2^4 - 1$ e) $1111^2 - 111^2$ f) $2^8 - 25$

Factorising quadratic expressions

$x^2 + 5x + 6$ is known as a quadratic expression as the highest power of any of its terms is squared, in this case x^2.

It can be factorised by writing it as a product of two brackets.

Worked examples **a)** Factorise $x^2 + 5x + 6$.

	x	
x	x^2	
		$+6$

On setting up a 2 × 2 grid, some of the information can immediately be entered.

As there is only one term in x^2, this can be entered, as can the constant +6. The only two values which multiply to give x^2 are x and x. These too can be entered.

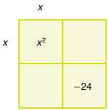
We now need to find two values which multiply to give +6 and which add to give $+5x$.

The only two values which satisfy both these conditions are +3 and +2.

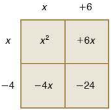
	x	$+3$
x	x^2	$3x$
$+2$	$2x$	$+6$

Therefore $x^2 + 5x + 6 = (x + 3)(x + 2)$.

b) Factorise $x^2 + 2x - 24$.



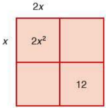
	x	
x	x^2	
		-24



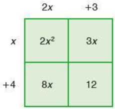
	x	$+6$
x	x^2	$+6x$
-4	$-4x$	-24

Therefore $x^2 + 2x - 24 = (x + 6)(x - 4)$.

c) Factorise $2x^2 + 11x + 12$.



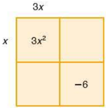
	$2x$	
x	$2x^2$	
		12



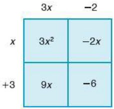
	$2x$	$+3$
x	$2x^2$	$3x$
$+4$	$8x$	12

Therefore $2x^2 + 11x + 12 = (2x + 3)(x + 4)$.

d) Factorise $3x^2 + 7x - 6$.



	$3x$	
x	$3x^2$	
		-6



	$3x$	-2
x	$3x^2$	$-2x$
$+3$	$9x$	-6

Therefore $3x^2 + 7x - 6 = (3x - 2)(x + 3)$.

Exercise 2.10

Factorise the following quadratic expressions:

1. a) $x^2 + 7x + 12$ b) $x^2 + 8x + 12$ c) $x^2 + 13x + 12$
d) $x^2 - 7x + 12$ e) $x^2 - 8x + 12$ f) $x^2 - 13x + 12$

2. a) $x^2 + 6x + 5$ b) $x^2 + 6x + 8$ c) $x^2 + 6x + 9$
d) $x^2 + 10x + 25$ e) $x^2 + 22x + 121$ f) $x^2 - 13x + 42$

3. a) $x^2 + 14x + 24$ b) $x^2 + 11x + 24$ c) $x^2 - 10x + 24$
d) $x^2 + 15x + 36$ e) $x^2 + 20x + 36$ f) $x^2 - 12x + 36$

4. a) $x^2 + 2x - 15$ b) $x^2 - 2x - 15$ c) $x^2 + x - 12$
d) $x^2 - x - 12$ e) $x^2 + 4x - 12$ f) $x^2 - 15x + 36$

5. a) $x^2 - 2x - 8$ b) $x^2 - x - 20$ c) $x^2 + x - 30$
d) $x^2 - x - 42$ e) $x^2 - 2x - 63$ f) $x^2 + 3x - 54$

6. a) $2x^2 + 4x + 2$ b) $2x^2 + 7x + 6$ c) $2x^2 + x - 6$
d) $2x^2 - 7x + 6$ e) $3x^2 + 8x + 4$ f) $3x^2 + 11x - 4$
g) $4x^2 + 12x + 9$ h) $9x^2 - 6x + 1$ i) $6x^2 - x - 1$

Rearranging complex formulae

Worked examples Make the letters in **red** the subject of each formula:

a) $C = 2\pi r$

$\frac{C}{2\pi} = r$

b) $A = \pi r^2$

$\frac{A}{\pi} = r^2$

$\sqrt{\frac{A}{\pi}} = r$

c) $Rx^2 = p$

$x^2 = \frac{p}{R}$

$x = \quad \frac{P}{R}$

d) $x^2 + y^2 = h^2$

$y^2 = h^2 - x^2$

$y = \sqrt{h^2 - x^2}$

Note: not
$y = h - x$

e) $\sqrt{x} = tv$

$x = t^2v^2$

or $x = (tv)^2$

f) $f = \sqrt{\frac{x}{k}}$

$f^2 = \frac{x}{k}$

$f^2k = x$

g) $m = 3a\sqrt{\frac{p}{x}}$

Square
both sides

$m^2 = \frac{9a^2p}{x}$

$m^2x = 9a^2p$

$x = \frac{9a^2p}{m^2}$

h) $A = \frac{y + x}{p + q^2}$

$A(p + q^2) = y + x$

$p + q^2 = \frac{y + x}{A}$

$q^2 = \frac{y + x}{A} - p$

$q = \sqrt{\frac{y + x}{A} - p}$

Exercise 2.11

In the formulae below, make x the subject:

1. a) $P = 2mx$ b) $T = 3x^2$
c) $mx^2 = y^2$ d) $x^2 + y^2 = p^2 - q^2$
e) $m^2 + x^2 = y^2 - n^2$ f) $p^2 - q^2 = 4x^2 - y^2$

2. a) $\frac{P}{Q} = rx$ b) $\frac{P}{Q} = rx^2$ c) $\frac{P}{Q} = \frac{x^2}{r}$
d) $\frac{m}{n} = \frac{1}{x^2}$ e) $\frac{r}{st} = \frac{w}{x^2}$ f) $\frac{p + q}{r} = \frac{w}{x^2}$

3. a) $\sqrt{x} = rp$ b) $\frac{mn}{p} = \sqrt{x}$

c) $g = \sqrt{\frac{k}{x}}$ d) $r = 2\pi\sqrt{\frac{x}{g}}$

e) $p^2 = \frac{4m^2r}{x}$ f) $p = 2m\sqrt{\frac{r}{x}}$

Exercise 2.12 In the following questions, make the letter in **red** the subject of the formula:

1. a) $v = u + a{\color{red}\boldsymbol{t}}$ b) $v^2 = {\color{red}\boldsymbol{u}}^2 + 2as$ c) $v^2 = u^2 + 2a{\color{red}\boldsymbol{s}}$

d) $s = {\color{red}\boldsymbol{u}}t + \frac{1}{2}at^2$ e) $s = ut + \frac{1}{2}{\color{red}\boldsymbol{a}}t^2$ f) $s = ut + \frac{1}{2}a{\color{red}\boldsymbol{t}}^2$

2. a) $A = \pi{\color{red}\boldsymbol{r}}\sqrt{s^2 + t^2}$ b) $A = \pi r\sqrt{{\color{red}\boldsymbol{h}}^2 + r^2}$

c) $\frac{1}{f} = \frac{1}{{\color{red}\boldsymbol{u}}} + \frac{1}{v}$ d) $\frac{1}{f} = \frac{1}{u} + \frac{1}{{\color{red}\boldsymbol{v}}}$

e) $t = 2\pi\sqrt{\frac{{\color{red}\boldsymbol{l}}}{g}}$ f) $t = 2\pi\sqrt{\frac{l}{{\color{red}\boldsymbol{g}}}}$

Algebraic fractions

Simplifying algebraic fractions

The rules for fractions involving algebraic terms are the same as those for numeric fractions. However the actual calculations are often easier when using algebra.

Worked examples

a) $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$ **b)** $\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}$

c) $\frac{\cancel{3}}{4} \times \frac{5}{\cancel{6}^{2}} = \frac{5}{8}$ **d)** $\frac{\cancel{a}}{c} \times \frac{b}{2\cancel{a}} = \frac{b}{2c}$

e) $\frac{\cancel{a}b}{e\cancel{c}} \times \frac{\cancel{c}d}{f\cancel{a}} = \frac{bd}{ef}$ **f)** $\frac{m^2}{m} = \frac{m \times \cancel{m}}{\cancel{m}} = m$

g) $\frac{x^5}{x^3} = \frac{\cancel{x} \times \cancel{x} \times \cancel{x} \times x \times x}{\cancel{x} \times \cancel{x} \times \cancel{x}} = x^2$

Exercise 2.13 Simplify the following algebraic fractions:

1. a) $\frac{x}{y} \times \frac{p}{q}$ b) $\frac{x}{y} \times \frac{q}{x}$ c) $\frac{p}{q} \times \frac{q}{r}$

d) $\frac{ab}{c} \times \frac{d}{ab}$ e) $\frac{ab}{c} \times \frac{d}{ac}$ f) $\frac{p^2}{q^2} \times \frac{q^2}{p}$

2. a) $\frac{m^3}{m}$ b) $\frac{r^7}{r^2}$ c) $\frac{x^9}{x^3}$

d) $\frac{x^2y^4}{xy^2}$ e) $\frac{a^2b^3c^4}{ab^2c}$ f) $\frac{pq^2r^4}{p^2q^3r}$

3. a) $\frac{4ax}{2ay}$ b) $\frac{12pq^2}{3p}$ c) $\frac{15mn^2}{3mn}$

d) $\frac{24x^5y^3}{8x^2y^2}$ e) $\frac{36p^2qr}{12pqr}$ f) $\frac{16m^2n}{24m^3n^2}$

4. a) $\frac{2}{b} \times \frac{a}{3}$ b) $\frac{4}{x} \times \frac{y}{2}$ c) $\frac{8}{x} \times \frac{x}{4}$

d) $\frac{9y}{2} \times \frac{2x}{3}$ e) $\frac{12x}{7} \times \frac{7}{4x}$ f) $\frac{4x^3}{3y} \times \frac{9y^2}{2x^2}$

5. a) $\frac{2ax}{3bx} \times \frac{4by}{a}$ b) $\frac{3p^2}{2q} \times \frac{5q}{3p}$

c) $\frac{p^2q}{rs} \times \frac{pr}{q}$ d) $\frac{a^2b}{fc^2} \times \frac{cd}{bd} \times \frac{ef^2}{ca^2}$

e) $\frac{2pq^2}{3rs} \times \frac{5m}{4q} \times \frac{8rs}{15p^2}$ f) $\frac{x^4}{wy^2} \times \frac{yz^2}{x^2} \times \frac{wx}{z^3}$

Addition and subtraction of algebraic fractions

In arithmetic it is easy to add or subtract fractions with the same denominator. It is the same process when dealing with algebraic fractions.

Worked examples

a) $\frac{4}{11} + \frac{3}{11}$
$= \frac{7}{11}$

b) $\frac{a}{11} + \frac{b}{11}$
$= \frac{a+b}{11}$

c) $\frac{4}{x} + \frac{3}{x}$
$= \frac{7}{x}$

If the denominators are different, the fractions need to be changed to form fractions with the same denominator.

d) $\frac{2}{9} + \frac{1}{3}$
$= \frac{2}{9} + \frac{3}{9}$
$= \frac{5}{9}$

e) $\frac{a}{9} + \frac{b}{3}$
$= \frac{a}{9} + \frac{3b}{9}$
$= \frac{a+3b}{9}$

f) $\frac{4}{5a} + \frac{7}{10a}$
$= \frac{8}{10a} + \frac{7}{10a}$
$= \frac{15}{10a}$
$= \frac{3}{2a}$

Similarly, with subtraction, the denominators need to be the same.

g) $\frac{7}{a} - \frac{1}{2a}$
$= \frac{14}{2a} - \frac{1}{2a}$
$= \frac{13}{2a}$

h) $\frac{p}{3} - \frac{q}{15}$
$= \frac{5p}{15} - \frac{q}{15}$
$= \frac{5p-q}{15}$

i) $\frac{5}{3b} - \frac{8}{9b}$
$= \frac{15}{9b} - \frac{8}{9b}$
$= \frac{7}{9b}$

Exercise 2.14

Simplify the following fractions:

1. a) $\frac{1}{7}+\frac{3}{7}$ b) $\frac{a}{7}+\frac{b}{7}$ c) $\frac{5}{13}+\frac{6}{13}$
d) $\frac{c}{13}+\frac{d}{13}$ e) $\frac{x}{3}+\frac{y}{3}+\frac{z}{3}$ f) $\frac{p^2}{5}+\frac{q^2}{5}$

2. a) $\frac{5}{11}-\frac{2}{11}$ b) $\frac{c}{11}-\frac{d}{11}$ c) $\frac{6}{a}-\frac{2}{a}$
d) $\frac{2a}{3}-\frac{5b}{3}$ e) $\frac{2x}{7}-\frac{3y}{7}$ f) $\frac{3}{4x}-\frac{5}{4x}$

3. a) $\frac{5}{6}-\frac{1}{3}$ b) $\frac{5}{2a}-\frac{1}{a}$ c) $\frac{2}{3c}+\frac{1}{c}$
d) $\frac{2}{x}+\frac{3}{2x}$ e) $\frac{5}{2p}-\frac{1}{p}$ f) $\frac{1}{w}-\frac{3}{2w}$

4. a) $\frac{p}{4}-\frac{q}{12}$ b) $\frac{x}{4}-\frac{y}{2}$ c) $\frac{m}{3}-\frac{n}{9}$
d) $\frac{x}{12}-\frac{y}{6}$ e) $\frac{r}{2}+\frac{m}{10}$ f) $\frac{s}{3}-\frac{t}{15}$

5. a) $\frac{3x}{4}-\frac{2x}{12}$ b) $\frac{3x}{5}-\frac{2y}{15}$ c) $\frac{3m}{7}+\frac{m}{14}$
d) $\frac{4m}{3p}-\frac{3m}{9p}$ e) $\frac{4x}{3y}-\frac{5x}{6y}$ f) $\frac{3r}{7s}+\frac{2r}{14s}$

Often one denominator is not a multiple of the other. In these cases the **lowest common multiple** of both denominators has to be found.

Worked examples

a) $\frac{1}{4}+\frac{1}{3}$
$=\frac{3}{12}+\frac{4}{12}$
$=\frac{7}{12}$

b) $\frac{1}{5}+\frac{2}{3}$
$=\frac{3}{15}+\frac{10}{15}$
$=\frac{13}{15}$

c) $\frac{a}{3}+\frac{b}{4}$
$=\frac{4a}{12}+\frac{3b}{12}$
$=\frac{4a+3b}{12}$

d) $\frac{2a}{3}+\frac{3b}{5}$
$=\frac{10a}{15}+\frac{9b}{15}$
$=\frac{10a+9b}{15}$

Exercise 2.15

Simplify the following fractions:

1. a) $\frac{1}{2}+\frac{1}{3}$ b) $\frac{1}{3}+\frac{1}{5}$ c) $\frac{1}{4}+\frac{1}{7}$
d) $\frac{2}{5}+\frac{1}{3}$ e) $\frac{1}{4}+\frac{5}{9}$ f) $\frac{2}{7}+\frac{2}{5}$

2. a) $\frac{a}{2}+\frac{b}{3}$ b) $\frac{a}{3}+\frac{b}{5}$ c) $\frac{p}{4}+\frac{q}{7}$
d) $\frac{2a}{5}+\frac{b}{3}$ e) $\frac{x}{4}+\frac{5y}{9}$ f) $\frac{2x}{7}+\frac{2y}{5}$

3. a) $\frac{a}{2}-\frac{a}{3}$ b) $\frac{a}{3}-\frac{a}{5}$ c) $\frac{p}{4}+\frac{p}{7}$
d) $\frac{2a}{5}+\frac{a}{3}$ e) $\frac{x}{4}+\frac{5x}{9}$ f) $\frac{2x}{7}+\frac{2x}{5}$

4. a) $\frac{3m}{5}-\frac{m}{2}$ b) $\frac{3r}{5}-\frac{r}{2}$ c) $\frac{5x}{4}-\frac{3x}{2}$
d) $\frac{2x}{7}+\frac{3x}{4}$ e) $\frac{11x}{2}-\frac{5x}{3}$ f) $\frac{2p}{3}-\frac{p}{2}$

5. a) $p-\frac{p}{2}$ b) $c-\frac{c}{3}$ c) $x-\frac{x}{5}$
d) $m-\frac{2m}{3}$ e) $q-\frac{4q}{5}$ f) $w-\frac{3w}{4}$

6. a) $2m-\frac{m}{2}$ b) $3m-\frac{2m}{3}$ c) $2m-\frac{5m}{2}$
d) $4m-\frac{3m}{2}$ e) $2p-\frac{5p}{3}$ f) $6q-\frac{6q}{7}$

7. a) $p-\frac{p}{r}$ b) $\frac{x}{y}+x$ c) $m+\frac{m}{n}$
d) $\frac{a}{b}+a$ e) $2x-\frac{x}{y}$ f) $2p-\frac{3p}{q}$

With more complex algebraic fractions, the method of getting a common denominator is still required.

Worked examples

a) $\frac{2}{x+1}+\frac{3}{x+2}$

$$= \frac{2(x+2)}{(x+1)(x+2)}+\frac{3(x+1)}{(x+1)(x+2)}$$

$$= \frac{2(x+2)+3(x+1)}{(x+1)(x+2)}$$

$$= \frac{2x+4+3x+3}{(x+1)(x+2)}$$

$$= \frac{5x+7}{(x+1)(x+2)}$$

b) $\dfrac{5}{p+3} - \dfrac{3}{p-5}$

$$= \frac{5(p-5)}{(p+3)(p-5)} - \frac{3(p+3)}{(p+3)(p-5)}$$

$$= \frac{5(p-5)-3(p+3)}{(p+3)(p-5)}$$

$$= \frac{5p-25-3p-9}{(p+3)(p-5)}$$

$$= \frac{2p-34}{(p+3)(p-5)}$$

c) $\dfrac{x^2-2x}{x^2+x-6}$

$$= \frac{x\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$$

$$= \frac{x}{x+3}$$

d) $\dfrac{x^2-3x}{x^2+2x-15}$

$$= \frac{x\cancel{(x-3)}}{\cancel{(x-3)}(x+5)}$$

$$= \frac{x}{x+5}$$

Exercise 2.16

Simplify the following algebraic fractions:

1. a) $\dfrac{1}{x+1} + \dfrac{2}{x+2}$ b) $\dfrac{3}{m+2} - \dfrac{2}{m-1}$

c) $\dfrac{2}{p-3} + \dfrac{1}{p-2}$ d) $\dfrac{3}{w-1} - \dfrac{2}{w+3}$

e) $\dfrac{4}{y+4} - \dfrac{1}{y+1}$ f) $\dfrac{2}{m-2} - \dfrac{3}{m+3}$

2. a) $\dfrac{x(x-4)}{(x-4)(x+2)}$ b) $\dfrac{y(y-3)}{(y+4)(y-3)}$

c) $\dfrac{(m+2)(m-2)}{(m-2)(m-3)}$ d) $\dfrac{p(p+5)}{(p-5)(p+5)}$

e) $\dfrac{m(2m+3)}{(m+4)(2m+3)}$ f) $\dfrac{(m+1)(m-1)}{(m+2)(m-1)}$

3. a) $\dfrac{x^2-5x}{(x+3)(x-5)}$ b) $\dfrac{x^2-3x}{(x+4)(x-3)}$

c) $\dfrac{y^2-7y}{(y-7)(y-1)}$ d) $\dfrac{x(x-1)}{x^2+2x-3}$

e) $\dfrac{x(x+2)}{x^2+4x+4}$ f) $\dfrac{x(x+4)}{x^2+5x+4}$

4. a) $\dfrac{x^2-x}{x^2-1}$ b) $\dfrac{x^2+2x}{x^2+5x+6}$

c) $\dfrac{x^2+4x}{x^2+x-12}$ d) $\dfrac{x^2-5x}{x^2-3x-10}$

e) $\dfrac{x^2+3x}{x^2-9}$ f) $\dfrac{x^2-7x}{x^2-49}$

SECTION 4

Linear and simultaneous equations

An equation is formed when the value of an unknown quantity is needed.

Simple linear equations

Worked examples Solve the following linear equations:

a) $3x + 8 = 14$
$3x = 6$
$x = 2$

b) $12 = 20 + 2x$
$-8 = 2x$
$-4 = x$

c) $3(p + 4) = 21$
$3p + 12 = 21$
$3p = 9$
$p = 3$

d) $4(x - 5) = 7(2x - 5)$
$4x - 20 = 14x - 35$
$4x + 15 = 14x$
$15 = 10x$
$1.5 = x$

Exercise 2.17 Solve the following linear equations:

1. a) $3x = 2x - 4$ b) $5y = 3y + 10$ c) $2y - 5 = 3y$
d) $p - 8 = 3p$ e) $3y - 8 = 2y$ f) $7x + 11 = 5x$

2. a) $3x - 9 = 4$ b) $4 = 3x - 11$
c) $6x - 15 = 3x + 3$ d) $4y + 5 = 3y - 3$
e) $8y - 31 = 13 - 3y$ f) $4m + 2 = 5m - 8$

3. a) $7m - 1 = 5m + 1$ b) $5p - 3 = 3 + 3p$
c) $12 - 2k = 16 + 2k$ d) $6x + 9 = 3x - 54$
e) $8 - 3x = 18 - 8x$ f) $2 - y = y - 4$

4. a) $\frac{x}{2} = 3$ b) $\frac{1}{2}y = 7$ c) $\frac{x}{4} = 1$
d) $\frac{1}{4}m = 3$ e) $7 = \frac{x}{5}$ f) $4 = \frac{1}{5}p$

5. a) $\frac{x}{3} - 1 = 4$ b) $\frac{x}{5} + 2 = 1$ c) $\frac{2}{3}x = 5$
d) $\frac{3}{4}x = 6$ e) $\frac{1}{5}x = \frac{1}{2}$ f) $\frac{2x}{5} = 4$

6. a) $\frac{x + 1}{2} = 3$ b) $4 = \frac{x - 2}{3}$ c) $\frac{x - 10}{3} = 4$
d) $8 = \frac{5x - 1}{3}$ e) $\frac{2(x - 5)}{3} = 2$ f) $\frac{3(x - 2)}{4} = 4x - 8$

7. a) $6 = \frac{2(y-1)}{3}$ b) $2(x+1) = 3(x-5)$

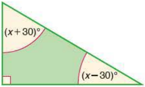
c) $5(x-4) = 3(x+2)$ d) $\frac{3+y}{2} = \frac{y+1}{4}$

e) $\frac{7+2x}{3} = \frac{9x-1}{7}$ f) $\frac{2x+3}{4} = \frac{4x-2}{6}$

Constructing simple equations

In many cases, when dealing with the practical applications of mathematics, equations need to be constructed first before they can be solved. Often the information is either given within the context of a problem or in a diagram.

Worked examples



a) Find the size of each of the angles in the triangle by constructing an equation and solving it to find the value of x.

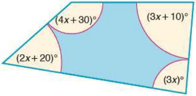
The sum of the angles of a triangle is 180°.

$$\begin{aligned}(x+30)+(x-30)+90 &= 180\\ 2x+90 &= 180\\ 2x &= 90\\ x &= 45\end{aligned}$$

The three angles are therefore: 90°, $x + 30 = 75°$, $x - 30 = 15°$.

Check: 90° + 75° + 15° = 180°.

b) Find the size of each of the angles in the quadrilateral by constructing an equation and solving it to find the value of x.

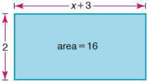


The sum of the angles of a quadrilateral is 360°.

$$\begin{aligned}4x+30+3x+10+3x+2x+20 &= 360\\ 12x+60 &= 360\\ 12x &= 300\\ x &= 25\end{aligned}$$

The angles are:

$$\begin{aligned}4x+30 &= (4\times25)+30 = 130°\\ 3x+10 &= (3\times25)+10 = 85°\\ 3x &= 3\times25 \quad = 75°\\ 2x+20 &= (2\times25)+20 = \underline{70°}\\ \text{Total} &= 360°\end{aligned}$$



c) Construct an equation and solve it to find the value of x in the diagram.

Area of rectangle = base × height

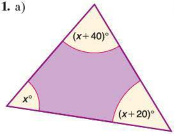
$$\begin{aligned} 2(x + 3) &= 16 \\ 2x + 6 &= 16 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

Exercise 2.18

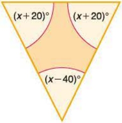
In Q 1–3:

i) construct an equation in terms of x
ii) solve the equation
iii) calculate the value of each of the angles
iv) check your answers.

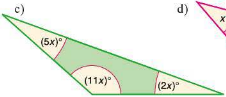
1. a)



b)



c)



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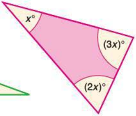

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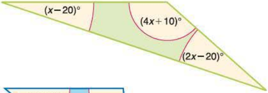
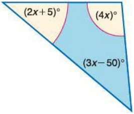
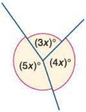


f)



2. a)



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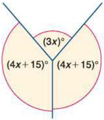

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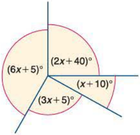

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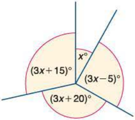
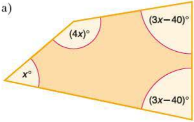


3. a)



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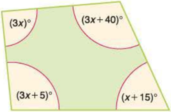

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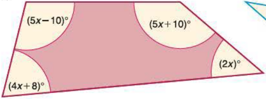

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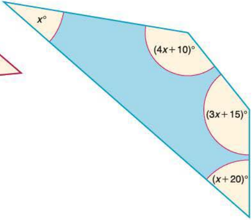

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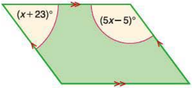
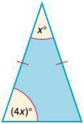


4. By constructing an equation and solving it, find the value of x in each of these isosceles triangles:

a)



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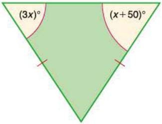
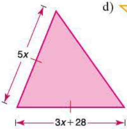
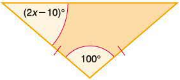


c)



d)



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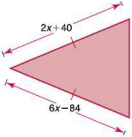

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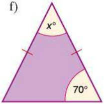
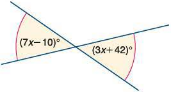


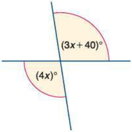
5. Using angle properties, calculate the value of *x* in each of these questions:

a)

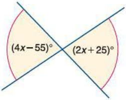
b)



c)

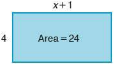


d)

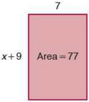


6. Calculate the value of *x*:

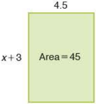
a)



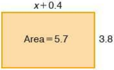
b)



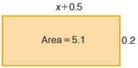
c)



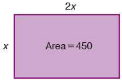
d)



e)



f)



Constructing formulae

$2x + 3 = 13$ is an equation. It is only true when $x = 5$.
$v = u + at$ is a formula because it describes the relationship between different variables which is true for all values of those variables. $v = u + at$ is a well-known formula for calculating the final velocity (v) of an object when its initial velocity (u), acceleration (a) and time taken (t) are known.

Worked examples

a) Using the formula $v = u + at$, calculate the final velocity (v) of a car in m/s if it started from rest and accelerated at a constant rate of $2\,\text{m/s}^2$ for 10 seconds.

$$\begin{aligned}\text{Therefore } u &= 0\,\text{m/s}\\ a &= 2\,\text{m/s}^2\\ t &= 10\,\text{s}\\ v &= 0 + (2 \times 10)\\ v &= 20\,\text{m/s}\end{aligned}$$

A formula can be rearranged to make different variables the subject of the formula.

b) Using the formula $v = u + at$ above, calculate the time it took for the car to reach a velocity of 30 m/s.

Rearrange the formula to make t the subject: $t = \frac{v-u}{a}$.

Therefore $t = \frac{30-0}{2} = 15\,\text{s}$.

It is important though also to be able to construct a formula from the information given.

c) Let T be the temperature in °C at the base of a mountain. It is known that the temperature falls by 1°C for each 200 m climbed.

i) Write a formula linking the temperature t (°C) at any point on the mountain to the height climbed h (m) and the base temperature T (°C).

$$t = T - \frac{h}{200}$$

ii) Calculate the temperature on the mountain at a height of 4000 m if the base temperature is 25 °C.

$$t = 25 - \frac{4000}{200} = 5$$

The temperature at a height of 4000 m is 5 °C.

iii) At what height does the temperature become 0 °C? Rearrange the formula to make h the subject:

$$h = 200(T - t)$$
$$h = 200(25 - 0) = 5000$$

The temperature is 0 °C at a height of 5000 m.

Exercise 2.19

1. The area (A) of a circle is given by the formula $A = \pi r^2$, where r is the radius.
   a) Calculate the area of a circle if its radius is 6.5 cm.
   b) Rearrange the formula to make r the subject.
   c) Calculate the radius of a circle with an area of 500 cm^2.

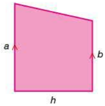
2. The volume (V) of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of its base and h its perpendicular height.
   a) Calculate the volume of a cone if $r = 8$ cm and $h = 18$ cm.
   b) A cone has a volume of 600 cm^3 and a height of 20 cm. Calculate the radius of its base.

3. To convert a temperature in °F (F) to °C (C) the following formula is used:

$$C = \tfrac{5}{9}(F - 32)$$

   a) Convert 100 °F to degrees Celsius.
   b) Rearrange the formula to make F the subject.
   c) Convert 10 °C to degrees Fahrenheit.

4. The distance, s (m), travelled by a moving object can be calculated using the formula $s = ut + \frac{1}{2}at^2$, where u represents the initial velocity (m/s), t is the time taken (s) and a is the acceleration (m/s^2).
   a) Calculate the distance a car will travel in 10 seconds if its initial velocity is 5 m/s and its acceleration is 1.5 m/s^2.
   b) Assuming the car starts from rest, rearrange the formula to make t the subject.
   c) Calculate the time taken for the car to travel 500 m if it starts from rest and $a = 1.5\ \text{m/s}^2$.



5. To calculate the area (A) of the trapezium shown, the mean length of the parallel sides a and b is calculated and then multiplied by the distance between them, h.
   a) Write a formula for calculating the area A of a trapezium.
   b) Calculate the area if $a = 9\,\text{cm}$, $b = 7\,\text{cm}$ and $h = 2.5\,\text{cm}$.
   c) If the area $A = 80\,\text{cm}^2$, $a = 20\,\text{cm}$ and $b = 12\,\text{cm}$, calculate the value of h.

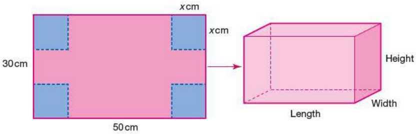
6. The cost C (€) of a taxi ride is €2.50/km plus a fixed charge of €5.
   a) Write a formula for calculating the cost of travelling n kilometres in the taxi.
   b) Rearrange the formula to make n the subject.
   c) A taxi journey cost €80. Calculate the length of the journey.

7. A bakery sells bread rolls for 20 cents each.
   a) Calculate the amount of change in dollars due if a customer buys 3 rolls and pays with $10.
   b) Write a formula for the amount of change given (C), when a $10 note is offered for x bread rolls.
   c) Write a formula for the amount of change given (C), when a $10 note is offered for x bread rolls costing y cents each.

8. A coffee shop sells three types of coffee, expresso, latte and cappucino. The cost of each are €e, €l and €c respectively.
   a) Write a formula for the total cost (T) of buying x expresso coffees.
   b) Write a formula for the total cost (T) of buying x expresso and y latte coffees.
   c) A customer buys x expresso, y latte and z cappuccino coffees. Write a formula to calculate the change due (C) if she pays with €20.

9. A dressmaker orders material online. The cost of the material is £15.50 per metre. The cost of delivery is £20 irrespective of the amount bought.
   a) Write a formula to calculate the total cost (C) of ordering n metres of material.
   b) i) Rearrange the formula to make n the subject.
      ii) If the total cost of ordering material came to £384.25, calculate the length of material ordered.

10. Metal containers are made by cutting squares from the corners of rectangular pieces of metal and are then folded as shown below.



The metal sheet has dimensions 50 cm × 30 cm. Squares of side length x cm are cut from each corner.

a) Write a formula in terms of x to calculate the length (L) of the container.
b) Write a formula in terms of x to calculate the width (W) of the container.
c) Write a formula in terms of x to calculate the height (H) of the container.
d) Write a formula in terms of x to calculate the volume (V) of the container.
e) Calculate the volume of the container, if a square of side length 12 cm is cut from each corner of the metal sheet.

Simultaneous equations

When the values of two unknowns are needed, two equations need to be formed and solved. The process of solving two equations and finding a common solution is known as solving equations simultaneously.

The two most common ways of solving simultaneous equations algebraically are by **elimination** and by **substitution**.

By elimination

The aim of this method is to eliminate one of the unknowns by either adding or subtracting the two equations.

Worked examples Solve the following simultaneous equations by finding the values of x and y which satisfy both equations:

a) $3x + y = 9$ (1)
$5x - y = 7$ (2)

By adding equations (1) + (2), we eliminate the variable y:

$$8x = 16$$
$$x = 2$$

To find the value of y we substitute $x = 2$ into either equation (1) or (2).

Substituting $x = 2$ into equation (1):

$$\begin{aligned} 3x + y &= 9 \\ 6 + y &= 9 \\ y &= 3 \end{aligned}$$

To check that the solution is correct, the values of x and y are substituted into equation (2). If it is correct then the left-hand side of the equation will equal the right-hand side.

$$\begin{aligned} 5x - y &= 7 \\ 10 - 3 &= 7 \\ 7 &= 7 \end{aligned}$$

b) $4x + y = 23$ (1)

$x + y = 8$ (2)

By subtracting the equations, i.e. (1) – (2), we eliminate the variable y:

$$\begin{aligned} 3x &= 15 \\ x &= 5 \end{aligned}$$

By substituting $x = 5$ into equation (2), y can be calculated:

$$\begin{aligned} x + y &= 8 \\ 5 + y &= 8 \\ y &= 3 \end{aligned}$$

Check by substituting both values into equation (1):

$$\begin{aligned} 4x + y &= 23 \\ 20 + 3 &= 23 \\ 23 &= 23 \end{aligned}$$

By substitution

The same equations can also be solved by the method known as **substitution**.

Worked example

a) $3x + y = 9$ (1)

$5x - y = 7$ (2)

Equation (2) can be rearranged to give: $y = 5x - 7$

This can now be substituted into equation (1):

$$\begin{aligned} 3x + (5x - 7) &= 9 \\ 3x + 5x - 7 &= 9 \\ 8x - 7 &= 9 \\ 8x &= 16 \\ x &= 2 \end{aligned}$$

To find the value of y, $x = 2$ is substituted into either equation (1) or (2) as before giving $y = 3$.

b) $4x + y = 23$ (1)
$x + y = 8$ (2)

Equation (2) can be rearranged to give $y = 8 - x$. This can be substituted into equation (1):

$$\begin{aligned} 4x + (8 - x) &= 23 \\ 4x + 8 - x &= 23 \\ 3x + 8 &= 23 \end{aligned} \qquad \begin{aligned} 3x &= 15 \\ x &= 5 \end{aligned}$$

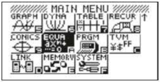
y can be found as before, giving a result of $y = 3$.

The Casio graphics calculator can solve simultaneous equations simply. Currently the Texas TI-84 Plus does not have this facility.

For example, to solve $3x + y = 9$ and $5x - y = 7$ simultaneously, use the following functions:

Casio

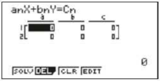
SET UP MENU A-LOCK ALPHA ∠ A X,θ,T to select the equation mode.



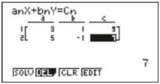
Trace F1 to select 'Simultaneous'.

Trace F1 to select the number of unknowns as 2.

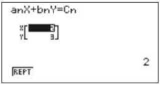
Enter the equation $3x + y = 9$ into the first row of the matrix where a is the coefficient of x, b is the coefficient of y and c the constant.



Enter the equation $5x - y = 7$ into the second row of the matrix.



Trace F1 to solve the simultaneous equations.



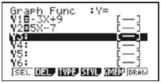
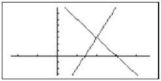
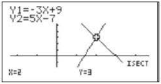
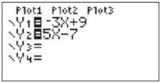
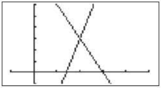
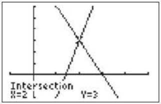
Note: The calculator requests the equations in the form $ax + by = c$.

Graphically, simultaneous equations are solved by plotting both lines on the same pair of axes and finding the coordinates of the point where the two lines intersect. Your calculator can solve simultaneous equations graphically.

For example, to solve the simultaneous equations $3x + y = 9$ and $5x - y = 7$ graphically, firstly rearrange each equation into gradient–intercept form, i.e. $y = mx + c$:

$$3x + y = 9 \rightarrow y = -3x + 9$$
$$5x - y = 7 \rightarrow y = 5x - 7$$

then use the following functions on your graphics calculator:

Casio	
MENU 5 to select the graphing mode. Enter the equations $y = -3x + 9$ and $y = 5x - 7$.	 Graph Func :Y= Y1■-3X+9 [—] Y2■5X-7 [—] Y3: [—] Y4: [—] Y5: [—] Y6: [—] SEL DEL TYPE STYL GMEM DRAW
F6 to graph both equations.	
SHIFT F5 F5 to find where the graphs intersect. The results are displayed on the screen.	 Y1=-3X+9 Y2=5X-7 ISECT X=2 Y=3
Texas	
Y= and enter the equations $y = -3x + 9$ and $y = 5x - 7$. GRAPH to graph both equations.	 Plot1 Plot2 Plot3 \Y1■-3X+9 \Y2■5X-7 \Y3= \Y4=
2ND TRACE 5 to calculate the coordinates of the point of intersection. Using the cursor select the first 'curve' then, when prompted, select the second 'curve'. Finally move the cursor over the point of intersection. ENTER The calculator will give the coordinates of the point of intersection.	  Intersection X=2 Y=3

Exercise 2.20

Solve the following simultaneous equations either by elimination or by substitution. Check your answers with a graphics calculator.

1. a) $x + y = 6$, $x - y = 2$
b) $x + y = 11$, $x - y - 1 = 0$
c) $x + y = 5$, $x - y = 7$
d) $2x + y = 12$, $2x - y = 8$
e) $3x + y = 17$, $3x - y = 13$
f) $5x + y = 29$, $5x - y = 11$

2. a) $3x + 2y = 13$, $4x = 2y + 8$
b) $6x + 5y = 62$, $4x - 5y = 8$
c) $x + 2y = 3$, $8x - 2y = 6$
d) $9x + 3y = 24$, $x - 3y = -14$
e) $7x - y = -3$, $4x + y = 14$
f) $3x = 5y + 14$, $6x + 5y = 58$

3. a) $2x + y = 14$, $x + y = 9$
b) $5x + 3y = 29$, $x + 3y = 13$
c) $4x + 2y = 50$, $x + 2y = 20$
d) $x + y = 10$, $3x = -y + 22$
e) $2x + 5y = 28$, $4x + 5y = 36$
f) $x + 6y = -2$, $3x + 6y = 18$

4. a) $x - y = 1$, $2x - y = 6$
b) $3x - 2y = 8$, $2x - 2y = 4$
c) $7x - 3y = 26$, $2x - 3y = 1$
d) $x = y + 7$, $3x - y = 17$
e) $8x - 2y = -2$, $3x - 2y = -7$
f) $4x - y = -9$, $7x - y = -18$

5. a) $x + y = -7$, $x - y = -3$
b) $2x + 3y = -18$, $2x = 3y + 6$
c) $5x - 3y = 9$, $2x + 3y = 19$
d) $7x + 4y = 42$, $9x - 4y = -10$
e) $4x - 4y = 0$, $8x + 4y = 12$
f) $x - 3y = -25$, $5x - 3y = -17$

6. a) $2x + 3y = 13$, $2x - 4y + 8 = 0$
b) $2x + 4y = 50$, $2x + y = 20$
c) $x + y = 10$, $3y = 22 - x$
d) $5x + 2y = 28$, $5x + 4y = 36$

7. a) $-4x = 4y$, $4x - 8y = 12$
b) $3x = 19 + 2y$, $-3x + 5y = 5$
c) $3x + 2y = 12$, $-3x + 9y = -12$
d) $3x + 5y = 29$, $3x + y = 13$
e) $-5x + 3y = 14$, $5x + 6y = 58$
f) $-2x + 8y = 6$, $2x = 3 - y$

If neither x nor y can be eliminated by simply adding or subtracting the two equations, then it is necessary to multiply one or both of the equations. The equations are multiplied by a number in order to make the coefficients of x (or y) numerically equal.

Worked examples

a)

$$\begin{aligned}3x + 2y &= 22 && (1)\\ x + y &= 9 && (2)\end{aligned}$$

Multiply equation (2) by 2 to eliminate y:

$$\begin{aligned}3x + 2y &= 22 && (1)\\ 2x + 2y &= 18 && (3)\end{aligned}$$

Subtract (3) from (1) to eliminate the variable y:

$$x = 4$$

Substitute $x = 4$ into equation (2):

$$\begin{aligned}x + y &= 9\\ 4 + y &= 9\\ y &= 5\end{aligned}$$

Check by substituting both values into equation (1):

$$\begin{aligned}3x + 2y &= 22\\ 12 + 10 &= 22\\ 22 &= 22\end{aligned}$$

b)

$$\begin{aligned}5x - 3y &= 1 && (1)\\ 3x + 4y &= 18 && (2)\end{aligned}$$

Multiply equation (1) by 4 and equation (2) by 3 to eliminate the variable y:

$$\begin{aligned}20x - 12y &= 4 && (3)\\ 9x + 12y &= 54 && (4)\end{aligned}$$

Add equations (3) and (4) to eliminate the variable y:

$$\begin{aligned}29x &= 58\\ x &= 2\end{aligned}$$

Substitute $x = 2$ into equation (2):

$$\begin{aligned}3x + 4y &= 18\\ 6 + 4y &= 18\\ 4y &= 12\\ y &= 3\end{aligned}$$

Check by substituting both values into equation (1):

$$\begin{aligned}5x - 3y &= 1\\ 10 - 9 &= 1\\ 1 &= 1\end{aligned}$$

Exercise 2.21

Solve the following simultaneous equations:

1. a) $2x + y = 7$
$3x + 2y = 12$

b) $5x + 4y = 21$
$x + 2y = 9$

c) $x + y = 7$
$3x + 4y = 23$

d) $2x - 3y = -3$
$3x + 2y = 15$

e) $4x = 4y + 8$
$x + 3y = 10$

f) $x + 5y = 11$
$2x - 2y = 10$

2. a) $x + y = 5$
$3x - 2y + 5 = 0$

b) $2x - 2y = 6$
$x - 5y = -5$

c) $2x + 3y = 15$
$2y = 15 - 3x$

d) $x - 6y = 0$
$3x - 3y = 15$

e) $2x - 5y = -11$
$3x + 4y = 18$

f) $x + y = 5$
$2x - 2y = -2$

3. a) $3y = 9 + 2x$
$3x + 2y = 6$

b) $x + 4y = 13$
$3x - 3y = 9$

c) $2x = 3y - 19$
$3x + 2y = 17$

d) $2x - 5y = -8$
$-3x - 2y = -26$

e) $5x - 2y = 0$
$2x + 5y = 29$

f) $8y = 3 - x$
$3x - 2y = 9$

4. a) $4x + 2y = 5$
$3x + 6y = 6$

b) $4x + y = 14$
$6x - 3y = 3$

c) $10x - y = -2$
$-15x + 3y = 9$

d) $-2y = 0.5 - 2x$
$6x + 3y = 6$

e) $x + 3y = 6$
$2x - 9y = 7$

f) $5x - 3y = -0.5$
$3x + 2y = 3.5$

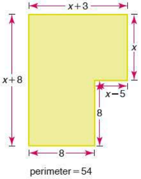
Constructing more complex equations

Earlier in this section we looked at some simple examples of constructing and solving equations when we were given geometrical diagrams. Here we extend this work with more complicated formulae and equations.

Worked examples

Construct and solve the equations below:

a) Using the shape below, construct an equation for the perimeter in terms of x. Find the value of x by solving the equation.



$$\begin{aligned} x + 3 + x + x - 5 + 8 + 8 + x + 8 &= 54 \\ 4x + 22 &= 54 \\ 4x &= 32 \\ x &= 8 \end{aligned}$$

b) A number is doubled, 5 is subtracted from the result, and the total is 17. Find the number.

Let x be the unknown number.

$$\begin{aligned} 2x - 5 &= 17 \\ 2x &= 22 \\ x &= 11 \end{aligned}$$

c) 3 is added to a number. The result is multiplied by 8. If the answer is 64, calculate the value of the original number.

Let x be the unknown number.

$$\begin{aligned} 8(x + 3) &= 64 \\ 8x + 24 &= 64 \\ 8x &= 40 \\ x &= 5 \end{aligned}$$

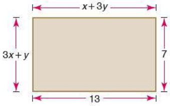
or $$\begin{aligned} 8(x + 3) &= 64 \\ x + 3 &= 8 \\ x &= 5 \end{aligned}$$

Exercise 2.22

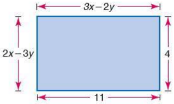
1. The sum of two numbers is 17 and their difference is 3. Find the two numbers by forming two equations and solving them simultaneously.

2. The difference between two numbers is 7. If their sum is 25, find the two numbers by forming two equations and solving them simultaneously.

3. Find the values of x and y:



4. Find the values of x and y:



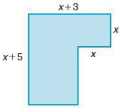
5. A man's age is three times his son's age. Ten years ago he was five times his son's age. By forming two equations and solving them simultaneously, find both of their ages.

6. A grandfather is ten times older than his granddaughter. He is also 54 years older than her. How old is each of them?

Exercise 2.23

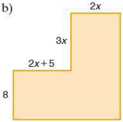
1. Calculate the value of x:

a)



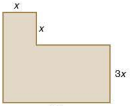
perimeter = 44

b)



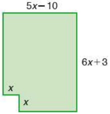
perimeter = 68

c)



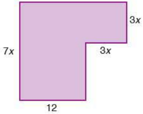
perimeter = 108

d)



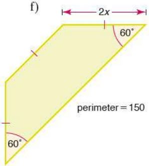
perimeter = 140

e)



perimeter = 224

f)



2. a) A number is trebled and then 7 is added to it. If the total is 28, find the number.
   b) Multiply a number by 4 and then add 5 to it. If the total is 29, find the number.
   c) If 31 is the result of adding 1 to 5 times a number, find the number.
   d) Double a number and then subtract 9. If the answer is 11, what is the number?
   e) If 9 is the result of subtracting 12 from 7 times a number, find the number.

3. a) Add 3 to a number and then double the result. If the total is 22, find the number.
   b) 27 is the answer when you add 4 to a number and then treble it. What is the number?
   c) Subtract 1 from a number and multiply the result by 5. If the answer is 35, what is the number?
   d) Add 3 to a number. If the result of multiplying this total by 7 is 63, find the number.
   e) Add 3 to a number. Quadruple the result. If the answer is 36, what is the number?

4. a) Gabriella is x years old. Her brother is 8 years older and her sister is 12 years younger than she is. If their total age is 50 years, how old is each of them?
   b) A series of Mathematics textbooks consists of four volumes. The first volume has x pages, the second 54 pages more. The third and fourth volume each have 32 pages more than the second. If the total number of pages in all four volumes is 866, calculate the number of pages in each of the volumes.
   c) The five interior angles (in °) of a pentagon are x, $x + 30$, $2x$, $2x + 40$ and $3x + 20$. The sum of the interior angles of a pentagon is 540°. Calculate the size of each of the angles.
   d) A hexagon consists of three interior angles of equal size and a further three which are double this size. The sum of all six angles is 720°. Calculate the size of each of the angles.
   e) Four of the exterior angles of an octagon are the same size. The other four are twice as big. If the sum of the exterior angles is 360°, calculate the size of the interior angles.

SECTION 5

Solving quadratic equations

An equation of the form $y = ax^2 + bx + c$, in which the highest power of the variable x is x^2, is known as a **quadratic equation**. The following are all types of quadratic equations:

$$y = x^2 + 2x - 4 \qquad y = -3x^2 + x + 2 \qquad y = x^2 \qquad y = \tfrac{1}{2}x^2 + 2$$

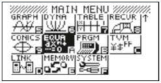
There are a number of ways to solve quadratic equations and the method used is largely dependent on the type of quadratic equation given. The main methods are explained later in this section, however you can also use your graphics calculator to solve quadratic equations and therefore check your answers.

Note: You will be given no credit in an exam for just writing down the answer to a quadratic equation problem, therefore you should use your calculator only as a tool for checking your answer.

Worked example Using your graphics calculator, solve the quadratic equation $6x^2 + 5x - 4 = 0$.

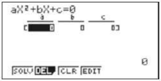
Casio

SET UP **MENU** A-LOCK **ALPHA** ∠ A **X,θ,T** to select the equation mode from the menu screen.

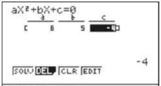


Zoom **F2** to select 'Polynomial'.

Trace **F1** to select the degree of the polynomial as 2.

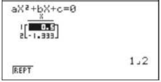


Enter the expression $6x^2 + 5x - 4$ into the matrix where a is the coefficient of x^2, b is the coefficient of x and c the constant.



Trace **F1** to solve the equation.

The solutions are given as $x = 0.5$ and -1.333.



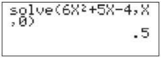
Note: The calculator requests the quadratic equation in the form $ax^2 + bx + c = 0$.
The solutions are given as a decimal by the calculator, but it is good practice to give your solution as a fraction, i.e. $x = \frac{1}{2}$ and $-\frac{4}{3}$.

Texas

2ND 0 to select 'catalog'. This displays all the operations possible in the TI-84. Scroll down to find 'solve(' ENTER

```
CATALOG          A
 sinh(
 sinh⁻¹(
 SinReg
▸solve(
 SortA(
```

Enter $6x^2 + 5x - 4, x, 0$ ENTER. This indicates the equation to be solved, the variable as x and the solution nearest to 0. The answer 0.5 is displayed.

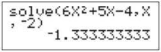


```
solve(6X²+5X-4,X
,0)
                .5
```

To find the second solution enter:

solve($6x^2 + 5x - 4, x, -2$ ENTER. This finds the solution nearest to −2.

The answer −1.3333 is displayed.



```
solve(6X²+5X-4,X
,-2)
       -1.333333333
```

Note: The Texas calculator is not as user-friendly for this operation. In order to use the solve facility when there is more than one solution, you have to know approximately where the solutions lie.

Scrolling through the catalog can be sped up by pressing 2ND 0 LN as this jumps to the letter 'S' in the catalog.

Solving quadratic equations by factorising

$x^2 - 3x - 10 = 0$ is a quadratic equation, which when factorised can be written as $(x - 5)(x + 2) = 0$.

Therefore either $x - 5 = 0$ or $x + 2 = 0$ since, if two things multiply to make zero, then one of them must be zero.

$$\begin{aligned} x - 5 &= 0 \quad \text{or} \quad x + 2 = 0 \\ x &= 5 \quad \text{or} \quad x = -2 \end{aligned}$$

Worked examples Solve the following equations to give two solutions for x:

a)

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$

so either $x - 4 = 0$ or $x + 3 = 0$

$x = 4$ or $x = -3$

b)

$$x^2 + 2x = 24$$

This becomes $x^2 + 2x - 24 = 0$

$$(x + 6)(x - 4) = 0$$

so either $x + 6 = 0$ or $x - 4 = 0$

$x = -6$ or $x = 4$

c) $x^2 - 6x = 0$

$x(x - 6) = 0$

so either $x = 0$ or $x - 6 = 0$

or $x = 6$

d) $x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

so either $x - 2 = 0$ or $x + 2 = 0$

$x = 2$ or $x = -2$

Exercise 2.24

Solve the following quadratic equations by factorising. Check your solutions using a calculator.

1. a) $x^2 + 7x + 12 = 0$ b) $x^2 + 8x + 12 = 0$
c) $x^2 + 13x + 12 = 0$ d) $x^2 - 7x + 10 = 0$
e) $x^2 - 5x + 6 = 0$ f) $x^2 - 6x + 8 = 0$

2. a) $x^2 + 3x - 10 = 0$ b) $x^2 - 3x - 10 = 0$
c) $x^2 + 5x - 14 = 0$ d) $x^2 - 5x - 14 = 0$
e) $x^2 + 2x - 15 = 0$ f) $x^2 - 2x - 15 = 0$

3. a) $x^2 + 5x = -6$ b) $x^2 + 6x = -9$
c) $x^2 + 11x = -24$ d) $x^2 - 10x = -24$
e) $x^2 + x = 12$ f) $x^2 - 4x = 12$

4. a) $x^2 - 2x = 8$ b) $x^2 - x = 20$
c) $x^2 + x = 30$ d) $x^2 - x = 42$
e) $x^2 - 2x = 63$ f) $x^2 + 3x = 54$

Exercise 2.25

Solve the following quadratic equations. Check your solutions using a calculator.

1. a) $x^2 - 9 = 0$ b) $x^2 - 16 = 0$
c) $x^2 = 25$ d) $x^2 = 121$
e) $x^2 - 144 = 0$ f) $x^2 - 220 = 5$

2. a) $4x^2 - 25 = 0$ b) $9x^2 - 36 = 0$
c) $25x^2 = 64$ d) $x^2 = \frac{1}{4}$
e) $x^2 - \frac{1}{9} = 0$ f) $16x^2 - \frac{1}{25} = 0$

3. a) $x^2 + 5x + 4 = 0$ b) $x^2 + 7x + 10 = 0$
c) $x^2 + 6x + 8 = 0$ d) $x^2 - 6x + 8 = 0$
e) $x^2 - 7x + 10 = 0$ f) $x^2 + 2x - 8 = 0$

4. a) $x^2 - 3x - 10 = 0$ b) $x^2 + 3x - 10 = 0$
c) $x^2 - 3x - 18 = 0$ d) $x^2 + 3x - 18 = 0$
e) $x^2 - 2x - 24 = 0$ f) $x^2 - 2x - 48 = 0$

5. a) $x^2 + x = 12$ b) $x^2 + 8x = -12$
c) $x^2 + 5x = 36$ d) $x^2 + 2x = -1$
e) $x^2 + 4x = -4$ f) $x^2 + 17x = -72$

6. a) $x^2 - 8x = 0$ b) $x^2 - 7x = 0$
c) $x^2 + 3x = 0$ d) $x^2 + 4x = 0$
e) $x^2 - 9x = 0$ f) $4x^2 - 16x = 0$

7. a) $2x^2 + 5x + 3 = 0$ b) $2x^2 - 3x - 5 = 0$
c) $3x^2 + 2x - 1 = 0$ d) $2x^2 + 11x + 5 = 0$
e) $2x^2 - 13x + 15 = 0$ f) $12x^2 + 10x - 8 = 0$

8. a) $x^2 + 12x = 0$ b) $x^2 + 12x + 27 = 0$
c) $x^2 + 4x = 32$ d) $x^2 + 5x = 14$
e) $2x^2 = 72$ f) $3x^2 - 12 = 288$

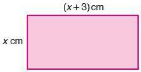
Exercise 2.26

In the following questions construct equations from the information given and then solve them to find the unknown.

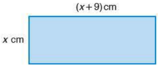
1. When a number x is added to its square, the total is 12. Find two possible values for x.

2. A number x is equal to its own square minus 42. Find two possible values for x.

3. If the area of the rectangle below is 10 cm², calculate the only possible value for x.



4. If the area of the rectangle is 52 cm², calculate the only possible value for x.



5. A triangle has a base length of $2x$ cm and a height of $(x - 3)$ cm. If its area is 18 cm², calculate its height and base length.

6. A triangle has a base length of $(x - 8)$ cm and a height of $2x$ cm. If its area is 20 cm², calculate its height and base length.

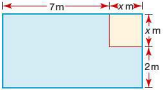
7. A right-angled triangle has a base length of x cm and a height of $(x - 1)$ cm. If its area is 15 cm², calculate the base length and height.

8. A rectangular garden has a square flowerbed of side length x m in one of its corners. The remainder of the garden consists of lawn and has dimensions as shown. If the total area of the lawn is $50\,\text{m}^2$:

a) form an equation in terms of x

b) solve the equation

c) calculate the length and width of the whole garden.



The quadratic formula

In general a quadratic equation takes the form $ax^2 + bx + c = 0$ where a, b and c are integers. Quadratic equations can be solved by the use of the quadratic formula which states that:

$$x = \frac{-b \pm \quad b^2 - 4ac}{2a}$$

Worked examples

a) Solve the quadratic equation $x^2 + 7x + 3 = 0$.

$a = 1$, $b = 7$ and $c = 3$

Substituting these values into the quadratic formula gives:

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 3}}{2 \times 1}$$

$$x = \frac{-7 \pm \sqrt{49 - 12}}{2}$$

$$x = \frac{-7 \pm \sqrt{37}}{2}$$

Therefore $x = \frac{-7 + 6.08}{2}$ or $x = \frac{-7 - 6.08}{2}$

$x = -0.46$ (2 d.p.) or $x = -6.54$ (2 d.p.)

b) Solve the quadratic equation $x^2 - 4x - 2 = 0$.

$a = 1$, $b = -4$ and $c = -2$

Substituting these values into the quadratic formula gives:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 1 -2)}}{2 \times 1}$$

$$x = \frac{4 \pm \sqrt{16 + 8}}{2}$$

$$x = \frac{4 \pm \sqrt{24}}{2}$$

$$\text{Therefore } x = \frac{4 + 4.90}{2} \quad \text{or} \quad x = \frac{4 - 4.90}{2}$$

$$x = 4.45 \text{ (2 d.p.)} \quad \text{or} \quad x = -0.45 \text{ (2 d.p.)}$$

Extension

Completing the square

Although the method of completing the square will not be assessed directly, this method often simplifies problems involving quadratics and their graphs.

Quadratics can also be solved by expressing them in terms of a perfect square. We look once again at the quadratic $x^2 - 4x - 2 = 0$.

The perfect square $(x - 2)^2$ can be expanded to give $x^2 - 4x + 4$. Notice that the x^2 and x terms are the same as those in the original quadratic.

Therefore $(x - 2)^2 - 6 = x^2 - 4x - 2$ and can be used to solve the quadratic.

$$\begin{aligned} (x-2)^2 - 6 &= 0 \\ (x-2)^2 &= 6 \\ x - 2 &= \pm\sqrt{6} \\ x &= 2 \pm \sqrt{6} \\ x = 4.45 \quad &\text{or} \quad x = -0.45 \end{aligned}$$

The quadratic formula stated earlier can be derived using the method of completing the square as shown:

Solve $ax^2 + bx + c = 0$.

Divide all through by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Complete the square: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Rearrange: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

Express both fractions with a common denominator of $4a^2$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

Simplify: $\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$

Take the square root of both sides: $x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$

Simplify: $x + \dfrac{b}{2a} = \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$

Rearrange: $x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

Simplify to give the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exercise 2.27

Solve the following quadratic equations using either the quadratic formula or by completing the square. Give your answers to 2 d.p.

1. a) $x^2 - x - 13 = 0$ b) $x^2 + 4x - 11 = 0$
c) $x^2 + 5x - 7 = 0$ d) $x^2 + 6x + 6 = 0$
e) $x^2 + 5x - 13 = 0$ f) $x^2 - 9x + 19 = 0$

2. a) $x^2 + 7x + 9 = 0$ b) $x^2 - 35 = 0$
c) $x^2 + 3x - 3 = 0$ d) $x^2 - 5x - 7 = 0$
e) $x^2 + x - 18 = 0$ f) $x^2 - 8 = 0$

3. a) $x^2 - 2x - 2 = 0$ b) $x^2 - 4x - 11 = 0$
c) $x^2 - x - 5 = 0$ d) $x^2 + 2x - 7 = 0$
e) $x^2 - 3x + 1 = 0$ f) $x^2 - 8x + 3 = 0$

4. a) $2x^2 - 3x - 4 = 0$ b) $4x^2 + 2x - 5 = 0$
c) $5x^2 - 8x + 1 = 0$ d) $-2x^2 - 5x - 2 = 0$
e) $3x^2 - 4x - 2 = 0$ f) $-7x^2 - x + 15 = 0$

SECTION 6

Using a graphics calculator to solve equations

As seen earlier, a linear equation, when plotted, gives a straight line.

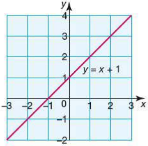
The following are all examples of linear equations:

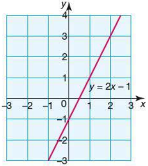
$y = x + 1 \qquad y = 2x - 1 \qquad y = 3x \qquad y = -x - 2 \qquad y = 4$

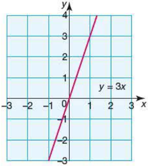
They all have a similar format, i.e. $y = mx + c$.

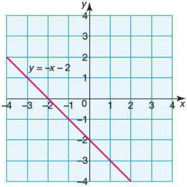
In the equation $y = x + 1$, $m = 1$ and $c = 1$
$y = 2x - 1$, $m = 2$ and $c = -1$
$y = 3x$, $m = 3$ and $c = 0$
$y = -x - 2$, $m = -1$ and $c = -2$
$y = 4$, $m = 0$ and $c = 4$

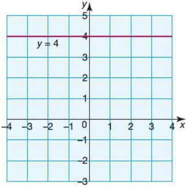
Their graphs are shown below:







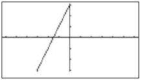




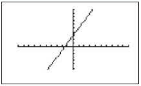
Using a graphics calculator to plot a linear equation

In the introductory topic you learned how to plot a single linear equation using your graphics calculator. For example, to graph the linear equation $y = 2x + 3$ (see overleaf).

Casio



Texas



Unless they are parallel to each other, when two linear graphs are plotted on the same axes, they will intersect at one point. Solving the equations simultaneously will give the coordinates of the point of intersection. Your graphics calculator will be able to work out the coordinates of the point of intersection.

Worked example Find the point of intersection of these linear equations:

$$y = 2x - 1 \text{ and } y = \tfrac{1}{2}x + 2$$

Casio

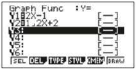
and enter $y = 2x - 1$,


Enter $y = \frac{1}{2}x + 2$,

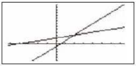

to graph the equations.

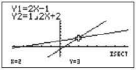
followed by


to select 'intersect' in the 'graph solve' menu. The results are displayed at the bottom of the screen.



```
Graph Func   :Y=
Y1■2X−1              [—]
Y2■1⌟2X+2            [—]
Y3:                  [—]
Y4:                  [—]
Y5:                  [—]
Y6:                  [—]
SEL DEL TYPE STYL MEM DRAW
```





```
Y1=2X−1
Y2=1⌟2X+2
                    ISECT
X=2        Y=3
```

Note: Equations of lines have to be entered in the form $y = \ldots$,
e.g. the equation $2x - 3y = 9$ would need to be rearranged to make
y the subject, i.e. $y = \frac{2x-9}{3}$ or $y = \frac{2}{3}x - 3$.

Texas

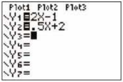
Y= and enter $y = 2x - 1$, **ENTER**

Then $y = \frac{1}{2}x + 2$, **ENTER**

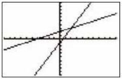
GRAPH to graph the equations.

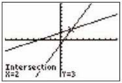
2ND **TRACE** followed by **5** to select 'intersect' in the 'graph calc' menu.

Once the two lines are selected using **ENTER**, the results are displayed at the bottom of the screen.



```
Plot1 Plot2 Plot3
\Y1■2X-1
\Y2■.5X+2
\Y3=■
\Y4=
\Y5=
\Y6=
\Y7=
```





```
Intersection
X=2          Y=3
```

Note: See the note for the Casio above.

Exercise 2.28

Use a graphics calculator to find the coordinates of the points of intersection of the following pairs of linear graphs:

1. a) $y = 5 - x$ and $y = x - 1$
   b) $y = 7 - x$ and $y = x - 3$
   c) $y = -2x + 5$ and $y = x - 1$
   d) $y = -x + 3$ and $y = 2x - 3$
   e) $x + 3y = -1$ and $y = \frac{1}{2}x + 3$
   f) $x - y = 6$ and $x + y = 2$

2. a) $3x - 2y = 13$ and $2x + y = 4$
   b) $4x - 5y = 1$ and $2x + y = -3$
   c) $x + 5 = y$ and $2x + 3y - 5 = 0$
   d) $x = y$ and $x + y + 6 = 0$
   e) $2x + y = 4$ and $4x + 2y = 8$
   f) $y - 3x = 1$ and $y = 3x - 3$

3. By referring to the lines, explain your answers to Q.2(e) and (f) above.

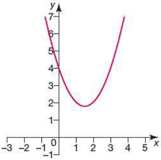
Quadratic equations

As you will recall from Section 5, an equation of the form $y = ax^2 + bx + c$, in which the highest power of the variable x is x^2, is known as a **quadratic equation**.

When plotted, a quadratic graph has a specific shape known as a **parabola**. This will look like  or 

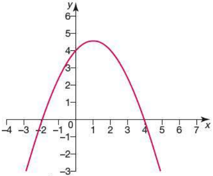
Depending on the values of a, b and c, the position and shape of the graph will vary, e.g.

$y = x^2 - 3x + 4$



$a = 1$, $b = -3$ and $c = 4$

$y = -\frac{1}{2}x^2 + x + 4$



$a = -\frac{1}{2}$, $b = 1$ and $c = 4$

Solving a quadratic equation of the form $ax^2 + bx + c = 0$ implies finding where the graph crosses the x-axis, because $y = 0$ on the x-axis.

In the case of $-\frac{1}{2}x^2 + x + 4 = 0$ above, it can be seen that the graph crosses the x-axis at $x = -2$ and $x = 4$. These are therefore the solutions to, or **roots** of, the equation.

In the case of $x^2 - 3x + 4 = 0$ above, the graph does not cross the x-axis. Therefore the equation has no real solutions. (Note: There are imaginary solutions, but these are not dealt with in this textbook.)

A graphics calculator can be used to find the solution to quadratic equations.

Casio

SET UP **MENU** Q **5** and enter $y = -\frac{1}{2}x^2 + x + 4$,

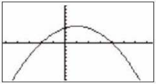
EXE

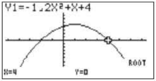
G↔T **F6** to graph the equations.

SHIFT G-Solv **F5** followed by Trace **F1** to select 'Root' in the 'graph solve' menu.

Use  to find the second root.

The results are displayed at the bottom of the screen.





```
Y1=-1⌟2X²+X+4
ROOT
X=4          Y=0
```

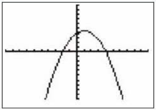
Texas

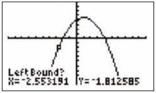
STAT PLOT F1 **Y=** and enter $y = -\frac{1}{2}x^2 + x + 4$, ENTRY SOLVE **ENTER**

TABLE F5 **GRAPH** to graph the equations.

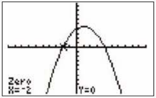
2ND CALC F4 **TRACE** followed by L2 Z **2** to select 'zero' in the 'graph calc' menu.

Use  and follow the on-screen prompts to identify a point to the left and a point to the right of the root, in order for the calculator to give the root.





```
LeftBound?
X=-2.553191  Y=-1.812585
```



```
Zero
X=-2         Y=0
```

Exercise 2.29 Using a graphics calculator:

i) graph the following quadratic equations
ii) find the coordinates of any roots.

1. a) $y = x^2 - 3x + 2$
b) $y = x^2 + 4x - 12$
c) $y = -x^2 + 8x - 15$
d) $y = x^2 + 2x + 6$
e) $y = -x^2 + x - 4$
f) $y = x^2 - 6x + 9$

2. a) $y = \frac{1}{2}x^2 - \frac{1}{2}x - 3$
b) $y = -2x^2 + 20x - 48$
c) $4y = -x^2 + 6x + 16$
d) $-2y = x^2 + 10x + 25$

SECTION 7

Linear inequalities

The statement

6 is less than 8 can be written as:

$6 < 8$

This inequality can be manipulated in the following ways:

adding 2 to each side:	$8 < 10$	this inequality is still true
subtracting 2 from each side:	$4 < 6$	this inequality is still true
multiplying both sides by 2:	$12 < 16$	this inequality is still true
dividing both sides by 2:	$3 < 4$	this inequality is still true
multiplying both sides by –2:	$-12 < -16$	this inequality is not true
dividing both sides by –2:	$-3 < -4$	this inequality is not true

As can be seen, when both sides of an inequality are either multiplied or divided by a negative number, the inequality is no longer true. For it to be true, the inequality sign needs to be changed around:

i.e. $-12 > -16$ and $-3 > -4$

The method used to solve linear inequalities is very similar to that used to solve linear equations.

Worked examples

Remember:

○→ implies that the number is not included in the solution. It is associated with $>$ and $<$.

●→ implies that the number is included in the solution. It is associated with $\geqslant$ and $\leqslant$.

Solve the following inequalities and represent the solution on a number line:

a) $15 + 3x < 6$

$3x < -9$

$x < -3$

b) $17 \leqslant 7x + 3$

$14 \leqslant 7x$

$2 \leqslant x$ that is $x \geqslant 2$

c) $9 - 4x \geqslant 17$

$-4x \geqslant 8$

$x \leqslant -2$

Note the inequality sign has changed direction.

Exercise 2.30

Solve the following inequalities and illustrate your solution on a number line:

1. a) $x + 3 < 7$ b) $5 + x > 6$
c) $4 + 2x \leqslant 10$ d) $8 \leqslant x + 1$
e) $5 > 3 + x$ f) $7 < 3 + 2x$

2. a) $x - 3 < 4$ b) $x - 6 \geqslant -8$
c) $8 + 3x > -1$ d) $5 \geqslant -x - 7$
e) $12 > -x - 12$ f) $4 \leqslant 2x + 10$

3. a) $\frac{x}{2} < 1$ b) $4 \geqslant \frac{x}{3}$
c) $1 \leqslant \frac{x}{2}$ d) $9x \geqslant -18$
e) $-4x + 1 < 3$ f) $1 \geqslant -3x + 7$

Worked example Find the range of values for which $7 < 3x + 1 \leqslant 13$ and illustrate the solutions on a number line.

This is in fact two inequalities which can therefore be solved separately.

$7 < 3x + 1$ and $3x + 1 \leqslant 13$

$6 < 3x$ $3x \leqslant 12$

$2 < x$ that is $x > 2$ $x \leqslant 4$

Exercise 2.31

Find the range of values for which the following inequalities are satisfied. Illustrate each solution on a number line:

1. a) $4 < 2x \leqslant 8$ b) $3 \leqslant 3x < 15$
c) $7 \leqslant 2x < 10$ d) $10 \leqslant 5x < 21$

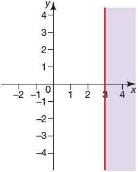
2. a) $5 < 3x + 2 \leqslant 17$ b) $3 \leqslant 2x + 5 < 7$
c) $12 < 8x - 4 < 20$ d) $15 \leqslant 3(x - 2) < 9$

The solution to an inequality can also be illustrated on a graph.

Worked examples **a)** On a pair of axes, shade the region which satisfies the inequality $x \geqslant 3$.

First draw the line $x = 3$.

Shade the region that represents the inequality $x \geqslant 3$, i.e. the region to the right of $x = 3$.

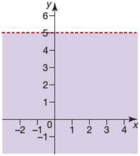


b) On a pair of axes, shade the region which satisfies the inequality $y < 5$.

First draw the line $y = 5$ (in this case it is drawn as a broken line).

Shade the region that represents the inequality $y < 5$, i.e. the region below the line $y = 5$.

Note: A broken (dashed) line shows $<$ or $>$ and a solid line shows $\leqslant$ or $\geqslant$.

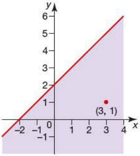


c) On a pair of axes, shade the region which satisfies the inequality $y \leqslant x + 2$.

First draw the line $y = x + 2$ (since it is included, this line is solid).

To find the region that satisfies the inequality, and hence to know which side of the line to shade:

1. Choose a point at random which does not lie on the line, e.g. (3, 1).
2. Substitute those values of x and y into the inequality, i.e. $1 \leqslant 3 + 2$.
3. If the inequality holds true, then the region in which the point lies satisfies the inequality and can therefore be shaded.



Note: In some questions the region which satisfies the inequality is left unshaded whilst in others it is shaded. You will therefore need to read the question carefully to see which is required.

Some graphics calculators can also be used to plot and shade the appropriate graphs of inequalities. The TI-84 does not currently have this facility. Guidance for the Casio is shown below.

For example, using worked example (c), shade, on a pair of axes, the region which satisfies the inequality $y \leqslant x + 2$.

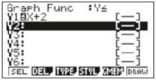
Casio

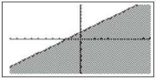
SET UP MENU Q 5 to select the graphing menu.

V-Window F3 G↔T F6 Sketch F4 to change the graph from Y= to Y≤.

Enter the inequality $y \leqslant x + 2$.

G↔T F6 to plot the inequality and shade the correct region.


Graph Func :Y≤
Y1≤X+2 [—]
Y2: [—]
Y3: [—]
Y4: [—]
Y5: [—]
Y6: [—]



Note: The calculator plot does not distinguish between the inequalities $y < x + 2$ and $y \leqslant x + 2$

Exercise 2.32

1. By drawing appropriate axes, shade the region which satisfies each of the following inequalities:
   - a) $y > 2$
   - b) $x < 3$
   - c) $y \leqslant 4$
   - d) $x \geqslant -1$
   - e) $y > 2x + 1$
   - f) $y \leqslant x - 3$
2. By drawing appropriate axes, leave unshaded the region which satisfies each of the following inequalities:
   - a) $y \geqslant -x$
   - b) $y \leqslant 2 - x$
   - c) $x \geqslant y - 3$
   - d) $x + y \geqslant 4$
   - e) $2x - y \geqslant 3$
   - f) $2y - x < 4$

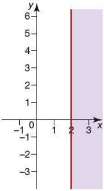
Several inequalities can be graphed on the same set of axes. If the regions which satisfy each inequality are left unshaded, then a solution can be found which satisfies all the inequalities, i.e. the region left unshaded by all the inequalities.

Worked example On the same pair of axes leave unshaded the regions which satisfy the following inequalities simultaneously:

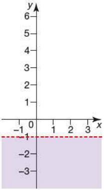
$$x \leqslant 2 \qquad y > -1 \qquad y \leqslant 3 \qquad y \leqslant x + 2$$

Hence find the region which satisfies all four inequalities.

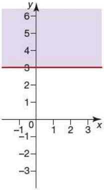
If the four inequalities are graphed on separate axes, the solutions are as shown on the next page:



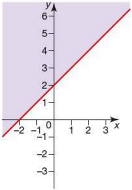
$x \leqslant 2$



$y > -1$

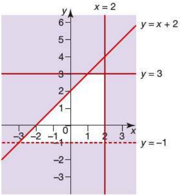


$y \leqslant 3$



$y \leqslant x + 2$

Combining all four on one pair of axes gives this diagram:



The unshaded region therefore gives a solution which satisfies all four inequalities.

Exercise 2.33

For Q.1-4, plot, on the same pair of axes, all the inequalities given. Leave unshaded the region which satisfies all of them simultaneously:

1. $y \leqslant x$ $\quad$ $y > 1$ $\quad$ $x \leqslant 5$

2. $x + y \leqslant 6$ $\quad$ $y < x$ $\quad$ $y \geqslant 1$

3. $y \geqslant 3x$ $\quad$ $y \leqslant 5$ $\quad$ $x + y > 4$

4. $2y \geqslant x + 4$ $\quad$ $y \leqslant 2x + 2$ $\quad$ $y < 4$ $\quad$ $x \leqslant 3$

Practical problems and inequalities

Inequalities are sometimes used to define problems. Solving the inequalities simultaneously can provide a number of possible solutions to the problem. More importantly, their solution can sometimes provide an optimum solution to the problem.

Worked example

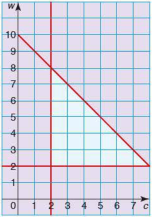
The number of fields a farmer plants with wheat is w and the number of fields he plants with corn is c. There are, however, certain restrictions which govern how many fields he can plant of each. These are as follows:

- There must be at least two fields of corn.
- There must be at least two fields of wheat.
- Not more than 10 fields can be sown in total.

i) Construct three inequalities from the information given above.

$c \geqslant 2$ $\quad$ $w \geqslant 2$ $\quad$ $c + w \leqslant 10$

ii) On one pair of axes, graph the three inequalities and leave unshaded the region which satisfies all three simultaneously.



iii) Give one possible arrangement as to how the farmer should plant his fields.

Four fields of corn and four fields of wheat.

The practical application of constructing and solving linear inequalities is sometimes called linear programming. More practice of this type of problem is given in Section 12.

SECTION 8

Laws of indices

The index refers to the power to which a number is raised. In the example 5^3, the number 5 is raised to the power 3. The 3 is known as the **index**. Indices is the plural of index.

Worked examples

a) $5^3 = 5 \times 5 \times 5$
$= 125$

b) $7^4 = 7 \times 7 \times 7 \times 7$
$= 2401$

c) $3^1 = 3$

Laws of indices

When working with numbers involving indices, there are three basic laws which can be applied. These are:

1. $a^m \times a^n = a^{m+n}$

2. $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$

3. $(a^m)^n = a^{mn}$

Positive indices

Worked examples

a) Simplify $4^3 \times 4^2$.
$4^3 \times 4^2 = 4^{(3+2)}$
$= 4^5$

b) Simplify $2^5 \div 2^3$.
$2^5 \div 2^3 = 2^{(5-3)}$
$= 2^2$

c) Evaluate $3^3 \times 3^4$.
$3^3 \times 3^4 = 3^{(3+4)}$
$= 3^7$
$= 2187$

d) Evaluate $(4^2)^3$.
$(4^2)^3 = 4^{(2\times3)}$
$= 4^6$
$= 4096$

Exercise 2.34

1. Using indices, simplify the following expressions:

a) $3 \times 3 \times 3$
b) $2 \times 2 \times 2 \times 2 \times 2$
c) 4×4
d) $6 \times 6 \times 6 \times 6$
e) $8 \times 8 \times 8 \times 8 \times 8 \times 8$
f) 5

2. Simplify the following using indices:

a) $2 \times 2 \times 2 \times 3 \times 3$
b) $4 \times 4 \times 4 \times 4 \times 4 \times 5 \times 5$
c) $3 \times 3 \times 4 \times 4 \times 4 \times 5 \times 5$
d) $2 \times 7 \times 7 \times 7 \times 7$
e) $1 \times 1 \times 6 \times 6$
f) $3 \times 3 \times 3 \times 4 \times 4 \times 6 \times 6 \times 6 \times 6 \times 6$

3. Write out the following in full:

a) 4^2
b) 5^7
c) 3^5
d) $4^3 \times 6^3$
e) $7^2 \times 2^7$
f) $3^2 \times 4^3 \times 2^4$

4. Without a calculator, work out the value of the following:

a) 2^5
b) 3^4
c) 8^2
d) 6^3
e) 10^6
f) 4^4
g) $2^3 \times 3^2$
h) $10^3 \times 5^3$

Exercise 2.35

1. Simplify the following using indices:

a) $3^2 \times 3^4$
b) $8^5 \times 8^2$
c) $5^2 \times 5^4 \times 5^3$
d) $4^3 \times 4^5 \times 4^2$
e) $2^1 \times 2^3$
f) $6^2 \times 3^2 \times 3^3 \times 6^4$
g) $4^5 \times 4^3 \times 5^5 \times 5^4 \times 6^2$
h) $2^4 \times 5^7 \times 5^3 \times 6^2 \times 6^6$

2. Simplify the following:

a) $4^6 \div 4^2$
b) $5^7 \div 5^4$
c) $2^5 \div 2^4$
d) $6^5 \div 6^2$
e) $\frac{6^5}{6^2}$
f) $\frac{8^6}{8^5}$
g) $\frac{4^8}{4^5}$
h) $\frac{3^9}{3^2}$

3. Simplify the following:

a) $(5^2)^2$
b) $(4^3)^4$
c) $(10^2)^5$
d) $(3^3)^5$
e) $(6^2)^4$
f) $(8^2)^3$

4. Simplify the following:

a) $\frac{2^2 \times 2^4}{2^3}$
b) $\frac{3^4 \times 3^2}{3^5}$
c) $\frac{5^6 \times 5^7}{5^2 \times 5^8}$
d) $\frac{(4^2)^5 \times 4^2}{4^7}$
e) $\frac{4^4 \times 2^5 \times 4^2}{4^3 \times 2^3}$
f) $\frac{6^3 \times 6^3 \times 8^5 \times 8^6}{8^6 \times 6^2}$
g) $\frac{(5^5)^2 \times (4^4)^3}{5^8 \times 4^9}$
h) $\frac{(6^3)^4 \times 6^3 \times 4^9}{6^8 \times (4^2)^4}$

5. Simplify the following:

a) $c^5 \times c^3$

b) $m^4 \div m^2$

c) $(b^3)^5 \div b^6$

d) $\dfrac{m^4n^9}{mn^3}$

e) $\dfrac{6a^6b^4}{3a^2b^3}$

f) $\dfrac{12x^5y^7}{4x^2y^5}$

g) $\dfrac{4u^3v^6}{8u^2v^3}$

h) $\dfrac{3x^6y^5z^3}{9x^4y^2z}$

6. Simplify the following:

a) $4a^2 \times 3a^3$

b) $2a^2b \times 4a^3b^2$

c) $(2p^2)^3$

d) $(4m^2n^3)^2$

e) $(5p^2)^2 \times (2p^3)^3$

f) $(4m^2n^2) \times (2mn^3)^3$

g) $\dfrac{(6x^2y^4)^2 \times (2xy)^3}{12x^6y^8}$

h) $(ab)^d \times (ab)^e$

The zero index

The zero index indicates that a number is raised to the power 0. A number raised to the power 0 is equal to 1. This can be explained by applying the laws of indices:

$a^m \div a^n = a^{m-n}$ therefore $\dfrac{a^m}{a^m} = a^{m-m}$

$= a^0$

However, $\dfrac{a^m}{a^m} = 1$

therefore $a^0 = 1$

Negative indices

A negative index indicates that a number is being raised to a negative power, e.g. 4^{-3}.

Another law of indices states that $a^{-m} = \dfrac{1}{a^m}$. It can be proved as follows:

$a^{-m} = a^{0-m}$

$= \dfrac{a^0}{a^m}$ (from the second law of indices)

$= \dfrac{1}{a^m}$

therefore $a^{-m} = \dfrac{1}{a^m}$

Exercise 2.36

Without using a calculator, evaluate the following:

1. a) $2^3 \times 2^0$ b) $5^2 \div 6^0$ c) $5^2 \times 5^{-2}$
d) $6^3 \times 6^{-3}$ e) $(4^0)^2$ f) $4^0 \div 2^2$

2. a) 4^{-1} b) 3^{-2} c) 6×10^{-2}
d) 5×10^{-3} e) 100×10^{-2} f) 10^{-3}

3. a) 9×3^{-2} b) 16×2^{-3} c) 64×2^{-4}
d) 4×2^{-3} e) 36×6^{-3} f) 100×10^{-1}

4. a) $\frac{3}{2^{-2}}$ b) $\frac{4}{2^{-3}}$ c) $\frac{9}{5^{-2}}$
d) $\frac{5}{4^{-2}}$ e) $\frac{7^{-3}}{7^{-4}}$ f) $\frac{8^{-6}}{8^{-8}}$

Exponential equations

Equations that involve indices as unknowns are known as **exponential equations**.

Worked examples

a) Find the value of x if $2^x = 32$.
Express 32 as a power of 2: $32 = 2^5$
Therefore $2^x = 2^5$
$x = 5$

b) Find the value of m if $3^m = 81$.
Express 81 as a power of 3: $81 = 3^4$
Therefore $3^m = 3^4$
$m = 4$

Exercise 2.37

1. Find the value of x in each of the following:
a) $2^x = 4$ b) $2^x = 16$
c) $4^x = 64$ d) $10^x = 1000$
e) $5^x = 625$ f) $3^x = 1$

2. Find the value of z in each of the following:
a) $2^{(z-1)} = 8$ b) $3^{(z+2)} = 27$
c) $4^{2z} = 64$ d) $10^{(z+1)} = 1$
e) $3^z = 9^{(z-1)}$ f) $5^z = 125$

3. Find the value of n in each of the following:
a) $(\frac{1}{2})^n = 8$ b) $(\frac{1}{3})^n = 81$
c) $(\frac{1}{2})^n = 32$ d) $(\frac{1}{2})^n = 4^{(n+1)}$
e) $(\frac{1}{2})^{(n+1)} = 2$ f) $(\frac{1}{16})^n = 4$

4. Find the value of x in each of the following:
a) $3^{-x} = 27$ b) $2^{-x} = 128$
c) $2^{(-x+3)} = 64$ d) $4^{-x} = \frac{1}{16}$
e) $2^{-x} = \frac{1}{256}$ f) $3^{(-x+1)} = \frac{1}{81}$

Exercise 2.38

1. A tap is dripping at a constant rate into a container. The level (l cm) of the water in the container, is given by the equation $l = 2^t - 1$ where t hours is the time taken.
   a) Calculate the level of the water after 3 hours.
   b) Calculate the level of the water in the container at the start.
   c) Calculate the time taken for the level of the water to reach 31 cm.
   d) Plot a graph showing the level of the water over the first 6 hours.
   e) From your graph, estimate the time taken for the water to reach a level of 45 cm.

2. Draw a graph of $y = 4^x$ for values of x between -1 and 3. Use your graph to find approximate solutions to these equations:
   a) $4^x = 30$
   b) $4^x = \frac{1}{2}$

3. Draw a graph of $y = 2^x$ for values of x between -2 and 5. Use your graph to find approximate solutions to the following equations:
   a) $2^x = 20$
   b) $2^{(x+2)} = 40$
   c) $2^{-x} = 0.2$

4. Draw a graph of $y = 3^x$ for values of x between -1 and 3. Use your graph to find approximate solutions to these equations:
   a) $3^{(x+2)} = 12$
   b) $3^{(x-3)} = 0.5$

SECTION 9

Fractional indices

Fractional indices

$16^{\frac{1}{2}}$ can be written as $(4^2)^{\frac{1}{2}}$.

$$
\begin{aligned}
(4^2)^{\frac{1}{2}} &= 4^{(2\times\frac{1}{2})} \\
&= 4^1 \\
&= 4
\end{aligned}
$$

Therefore $16^{\frac{1}{2}} = 4$

but $16 = 4$

therefore $16^{\frac{1}{2}} = 16$

Similarly:

$27^{\frac{1}{3}}$ can be written as $(3^3)^{\frac{1}{3}}$

$$(3^3)^{\frac{1}{3}} = 3^{\left(3 \times \frac{1}{3}\right)}$$
$$= 3^1$$
$$= 3$$

Therefore $27^{\frac{1}{3}} = 3$

but $\sqrt[3]{27} = 3$

therefore $27^{\frac{1}{3}} = \sqrt[3]{27}$

In general:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$
$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} \text{ or } \sqrt[n]{a^m}$$

Worked examples

a) Evaluate $16^{\frac{1}{4}}$ without the use of a calculator.

$$16^{\frac{1}{4}} = {}^{4}\ 16$$
$$= {}^{4}\ (2^4)$$
$$= 2$$

Alternatively: $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}}$
$$= 2^1$$
$$= 2$$

b) Evaluate $25^{\frac{3}{2}}$ without the use of a calculator.

$$25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3$$
$$= (\sqrt{25})^3$$
$$= 5^3$$
$$= 125$$

Alternatively: $25^{\frac{3}{2}} = (5^2)^{\frac{3}{2}}$
$$= 5^3$$
$$= 125$$

c) Solve $32^x = 2$

32 is 2^5 so $\sqrt[5]{32} = 2$

or $32^{\frac{1}{5}} = 2$

therefore $x = \frac{1}{5}$

d) Solve $125^x = 5$

125 is 5^3 so $\sqrt[3]{125} = 5$

or $125^{\frac{1}{3}} = 5$

therefore $x = \frac{1}{3}$

Exercise 2.39

For Q.1–4, evaluate all parts without the use of a calculator:

1. a) $16^{\frac{1}{2}}$ b) $25^{\frac{1}{2}}$ c) $100^{\frac{1}{2}}$
d) $27^{\frac{1}{3}}$ e) $81^{\frac{1}{2}}$ f) $1000^{\frac{1}{3}}$

2. a) $16^{\frac{1}{4}}$ b) $81^{\frac{1}{4}}$ c) $32^{\frac{1}{5}}$
d) $64^{\frac{1}{6}}$ e) $216^{\frac{1}{3}}$ f) $256^{\frac{1}{4}}$

3. a) $4^{\frac{3}{2}}$ b) $4^{\frac{5}{2}}$ c) $9^{\frac{3}{2}}$
d) $16^{\frac{3}{2}}$ e) $1^{\frac{5}{2}}$ f) $27^{\frac{2}{3}}$

4. a) $125^{\frac{2}{3}}$ b) $32^{\frac{3}{5}}$ c) $64^{\frac{5}{6}}$
d) $1000^{\frac{2}{3}}$ e) $16^{\frac{5}{4}}$ f) $81^{\frac{3}{4}}$

For Q.5-6, solve each equation without the use of a calculator:

5. a) $16^x = 4$ b) $8^x = 2$
c) $9^x = 3$ d) $27^x = 3$
e) $100^x = 10$ f) $64^x = 2$

6. a) $1000^x = 10$ b) $49^x = 7$
c) $81^x = 3$ d) $343^x = 7$
e) $1\,000\,000^x = 10$ f) $216^x = 6$

Exercise 2.40

Evaluate the following without the use of a calculator:

1. a) $\frac{27^{\frac{2}{3}}}{3^2}$ b) $\frac{7^{\frac{3}{2}}}{\sqrt{7}}$ c) $\frac{4^{\frac{5}{2}}}{4^2}$

d) $\frac{16^{\frac{3}{2}}}{2^6}$ e) $\frac{27^{\frac{5}{3}}}{\sqrt{9}}$ f) $\frac{6^{\frac{4}{3}}}{6^{\frac{1}{3}}}$

2. a) $5^{\frac{2}{3}} \times 5^{\frac{4}{3}}$ b) $4^{\frac{1}{4}} \times 4^{\frac{1}{4}}$ c) 8×2^{-2}
d) $3^{\frac{4}{3}} \times 3^{\frac{5}{3}}$ e) $2^{-2} \times 16$ f) $8^{\frac{5}{3}} \times 8^{-\frac{4}{3}}$

3. a) $\frac{2^{\frac{1}{2}} \times 2^{\frac{5}{2}}}{2}$ b) $\frac{4^{\frac{5}{6}} \times 4^{\frac{1}{6}}}{4^{\frac{1}{2}}}$ c) $\frac{2^3 \times 8^{\frac{3}{2}}}{\sqrt{8}}$

d) $\frac{(3^2)^{\frac{3}{2}} \times 3^{-\frac{1}{2}}}{3^{\frac{1}{2}}}$ e) $\frac{8^{\frac{1}{3}} + 7}{27^{\frac{1}{3}}}$ f) $\frac{9^{\frac{1}{2}} \times 3^{\frac{5}{2}}}{3^{\frac{2}{3}} \times 3^{-\frac{1}{6}}}$

SECTION 10 Sequences

A **sequence** is a collection of terms arranged in a specific order, where each term is obtained according to a rule. Examples of some simple sequences are given below:

2, 4, 6, 8, 10
1, 1, 2, 3, 5, 8
1, 4, 9, 16, 25
1, 8, 27, 64, 125
1, 2, 4, 8, 16
10, 5, $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$

You could discuss with another student the rules involved in producing the sequences above.

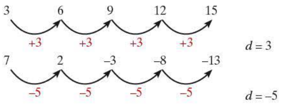
The terms of a sequence can be expressed as $u_1, u_2, u_3, \ldots, u_n$ where:

u_1 is the first term
u_2 is the second term
u_n is the nth term

Therefore in the sequence 2, 4, 6, 8, 10, $u_1 = 2$, $u_2 = 4$, etc.

Arithmetic sequences

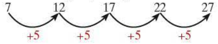
In an **arithmetic sequence** there is a common difference (d) between successive terms. Examples of some arithmetic sequences are given below:



Formulae for the terms of an arithmetic sequence

There are two main ways of describing a sequence.

1. A term-to-term rule
   In the following sequence,



the term-to-term rule is +5, i.e. $u_2 = u_1 + 5$, $u_3 = u_2 + 5$, etc. The general form is therefore written as $u_{n+1} = u_n + 5$, $u_1 = 7$, where u_n is the nth term and u_{n+1} the term after the nth term.

Note: It is important to give one of the terms, e.g. u_1, so that the exact sequence can be generated.

2. A formula for the nth term of a sequence
This type of rule links each term to its position in the sequence, e.g.

Position	1	2	3	4	5	n
Term	7	12	17	22	27	

We can deduce from the figures above that each term can be calculated by multiplying its position number by 5 and adding 2. Algebraically this can be written as the formula for the nth term:

$$u_n = 5n + 2$$

This textbook focuses on the generation and use of the rule for the nth term.

With an arithmetic sequence, the rule for the nth term can easily be deduced by looking at the common difference, e.g.

Position	1	2	3	4	5
Term	1	5	9	13	17

+4 +4 +4 +4

$u_n = 4n - 3$

Position	1	2	3	4	5
Term	7	9	11	13	15

+2 +2 +2 +2

$u_n = 2n + 5$

Position	1	2	3	4	5
Term	12	9	6	3	0

–3 –3 –3 –3

$u_n = -3n + 15$

The common difference is the coefficient of n (i.e. the number by which n is multiplied). The constant is then worked out by calculating the number needed to make the term.

Worked example Find the rule for the nth term of the sequence 12, 7, 2, –3, –8, …

Position	1	2	3	4	5
Term	12	7	2	–3	–8

–5 –5 –5 –5

$u_n = -5n + 17$

Exercise 2.41

1. For each of the following sequences:

i) deduce the formula for the nth term
ii) calculate the 10th term.

a) 5, 8, 11, 14, 17
b) 0, 4, 8, 12, 16
c) $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}$
d) 6, 3, 0, –3, –6
e) –7, –4, –1, 2, 5
f) –9, –13, –17, –21, –25

2. Copy and complete each of the following tables of arithmetic sequences:

a)

Position	1	2	5		50	n
Term				45		$4n - 3$

b)

Position	1	2	5			n
Term				59	449	$6n - 1$

c)

Position	1				100	n
Term		0	–5	–47		$-n + 3$

d)

Position	1	2	3			n
Term	3	0	–3	–24	–294	

e)

Position		5	7			n
Term	1	10	16	25	145	

f)

Position	1	2	5		50	n
Term	–5.5	–7		–34		

3. For each of the following arithmetic sequences:

i) deduce the common difference d
ii) give the formula for the nth term
iii) calculate the 50th term.

a) 5, 9, 13, 17, 21
b) 0, …, 2, …, 4
c) –10, …, …, …, 2
d) $u_1 = 6, u_9 = 10$
e) $u_3 = -50, u_{20} = 18$
f) $u_5 = 60, u_{12} = 39$

Sequences with quadratic and cubic rules

So far all the sequences we have looked at have been arithmetic, i.e. the rule for the *n*th term is linear and takes the form $u_n = an + b$. The rule for the *n*th term can be found algebraically using the method of differences and this method is particularly useful for more complex sequences.

Worked examples

a) Deduce the rule for the *n*th term for the sequence 4, 7, 10, 13, 16, … .

Firstly, produce a table of the terms and their positions in the sequence:

Position	1	2	3	4	5
Term	4	7	10	13	16

Extend the table to look at the differences:

Position	1		2		3		4		5
Term	4		7		10		13		16
1st Difference		3		3		3		3	

As the row of 1st differences is constant, the rule for the *n*th term is linear and takes the form $u_n = an + b$.

By substituting the values of n into the rule, each term can be expressed in terms of a and b:

Position	1		2		3		4		5
Term	$a + b$		$2a + b$		$3a + b$		$4a + b$		$5a + b$
1st Difference		a		a		a		a	

Compare the two tables in order to deduce the values of a and b:

$a = 3$
$a + b = 4$ therefore $b = 1$

The rule for the *n*th term $u_n = an + b$ can be written as $u_n = 3n + 1$.

For a linear rule, this method is perhaps overcomplicated. However it is very efficient for quadratic and cubic rules.

b) Deduce the rule for the nth term for the sequence 0, 7, 18, 33, 52, … .

Entering the sequence in a table gives:

Position	1	2	3	4	5
Term	0	7	18	33	52

Extending the table to look at the differences gives:

Position	1	2	3	4	5
Term	0	7	18	33	52
1st Difference		7	11	15	19

The row of 1st differences is not constant, and so the rule for the nth term is not linear. Extend the table again to look at the row of 2nd differences:

Position	1	2	3	4	5
Term	0	7	18	33	52
1st Difference		7	11	15	19
2nd Difference		4	4	4	

The row of 2nd differences is constant, and so the rule for the nth term is therefore a quadratic which takes the form $u_n = an^2 + bn + \mathrm{c}$.

By substituting the values of n into the rule, each term can be expressed in terms of a, b and c as shown:

Position	1	2	3	4	5
Term	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$	$25a + 5b + c$
1st Difference		$3a + b$	$5a + b$	$7a + b$	$9a + b$
2nd Difference		$2a$	$2a$	$2a$	

Comparing the two tables, the values of a, b and c can be deduced:

$2a = 4$ therefore $a = 2$

$3a + b = 7$ therefore $6 + b = 7$ giving $b = 1$

$a + b + c = 0$ therefore $2 + 1 + c = 0$ giving $c = -3$

The rule for the nth term $u_n = an^2 + bn + \mathrm{c}$ can be written as $u_n = 2n^2 + n - 3$.

c) Deduce the rule for the nth term for the sequence –6, –8, –6, 6, 34, … .

Entering the sequence in a table gives:

Position	1	2	3	4	5
Term	–6	–8	–6	6	34

Extending the table to look at the differences:

Position	1	2	3	4	5
Term	–6	–8	–6	6	34
1st Difference		–2	2	12	28

The row of 1st differences is not constant, and so the rule for the nth term is not linear. Extend the table again to look at the row of 2nd differences:

Position	1	2	3	4	5
Term	–6	–8	–6	6	34
1st Difference		–2	2	12	28
2nd Difference		4	10	16	

The row of 2nd differences is not constant either, and so the rule for the nth term is not quadratic. Extend the table by a further row to look at the row of 3rd differences:

Position	1	2	3	4	5
Term	–6	–8	–6	6	34
1st Difference		–2	2	12	28
2nd Difference		4	10	16	
3rd Difference			6	6	

The row of 3rd differences is constant, and so the rule for the nth term is therefore a cubic which takes the form $u_n = an^3 + bn^2 + cn + d$.

By substituting the values of n into the rule, each term can be expressed in terms of a, b, c, and d as shown:

Position	1	2	3	4	5
Term	$a + b + c + d$	$8a + 4b + 2c + d$	$27a + 9b + 3c + d$	$64a + 16b + 4c + d$	$125a + 25b + 5c + d$
1st Difference		$7a + 3b + c$	$19a + 5b + c$	$37a + 7b + c$	$61a + 9b + c$
2nd Difference		$12a + 2b$	$18a + 2b$	$24a + 2b$	
3rd Difference			$6a$	$6a$	

By comparing the two tables, equations can be formed and the values of *a, b, c*, and *d* can be found:

$6a = 6$ therefore $a = 1$

$12a + 2b = 4$ therefore $12 + 2b = 4$ giving $b = -4$

$7a + 3b + c = -2$ therefore $7 - 12 + c = -2$ giving $c = 3$

$a + b + c + d = -6$ therefore $1 - 4 + 3 + d = -6$ giving $d = -6$

Therefore the equation for the *n*th term is $u_n = n^3 - 4n^2 + 3n - 6$.

Exercise 2.42

By using a table if necessary, find the formula for the *n*th term of each of the following sequences:

1. 2, 5, 10, 17, 26
2. 0, 3, 8, 15, 24
3. 6, 9, 14, 21, 30
4. 9, 12, 17, 24, 33
5. –2, 1, 6, 13, 22
6. 4, 10, 20, 34, 52
7. 0, 6, 16, 30, 48
8. 5, 14, 29, 50, 77
9. 0, 12, 32, 60, 96
10. 1, 16, 41, 76, 121

Exercise 2.43

Use a table to find the formula for the *n*th term of the following sequences:

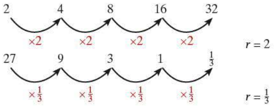
1. 11, 18, 37, 74, 135
2. 0, 6, 24, 60, 120
3. –4, 3, 22, 59, 120
4. 2, 12, 36, 80, 150
5. 7, 22, 51, 100, 175
6. 7, 28, 67, 130, 223
7. 1, 10, 33, 76, 145
8. 13, 25, 49, 91, 157

Geometric sequences

So far we have looked at sequences where there is a common difference between successive terms. There are, however, other types of sequences, e.g. 2, 4, 8, 16, 32. There is clearly a pattern to the way the numbers are generated as each term is double the previous term, but there is no common difference.

A sequence where there is a **common ratio** (r) between successive terms is known as a **geometric sequence**.

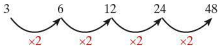
e.g.



As with an arithmetic sequence, there are two main ways of describing a geometric sequence.

1. The term-to-term rule

   For example, for the following sequence,



   $u_2 = 2u_1$ $u_3 = 2u_2$

   the general rule is $u_{n+1} = 2u_n$; $u_1 = 3$.

2. The formula for the nth term of a geometric sequence

   As with an arithmetic sequence, this rule links each term to its position in the sequence,

Position	1	2	3	4	5	n
Term	3	6	12	24	48	

   to reach the second term the calculation is 3×2 or 3×2^1
   to reach the third term, the calculation is $3 \times 2 \times 2$ or 3×2^2
   to reach the fourth term, the calculation is $3 \times 2 \times 2 \times 2$ or 3×2^3

   In general therefore

   $$u_n = ar^{n-1}$$

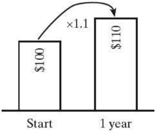
   where a is the first term and r is the common ratio.

Applications of geometric sequences

In Topic 1 simple and compound interest were shown as different ways that interest could be earned on money left in a bank account for a period of time. Here we look at compound interest as an example of a geometric sequence.

Compound interest

e.g. \$100 is deposited in a bank account and left untouched. After 1 year the amount has increased to \$110 as a result of interest payments. To work out the interest rate, calculate the multiplier from \$100 → \$110:



$\frac{110}{100} = 1.1$

The multiplier is 1.1. This corresponds to a 10% increase. Therefore the simple interest rate is 10% in the first year.

Assume the money is left in the account and that the interest rate remains unchanged. Calculate the amount in the account after 5 years.

This is an example of a geometric sequence.

Number of years	0	1	2	3	4	5
Amount	100.00	110.00	121.00	133.10	146.41	161.05
		×1.1	×1.1	×1.1	×1.1	×1.1

Alternatively the amount after 5 years can be calculated using a variation of $u_n = ar^{n-1}$, i.e. $u_5 = 100 \times 1.1^5 = 161.05$. Note: As the number of years starts at 0, ×1.1 is applied 5 times to get to the fifth year.

This is an example of compound interest as the previous year's interest is added to the total and included in the following year's calculation.

Worked examples

a) Alex deposits \$1500 in his savings account. The interest rate offered by the savings account is 6% each year for a 10-year period. Assuming Alex leaves the money in the account, calculate how much interest he has gained after the 10 years.

An interest rate of 6% implies a common ratio of 1.06

Therefore $u_{10} = 1500 \times 1.06^{10} = 2686.27$

The amount of interest gained is $2686.27 - 1500 = \$1186.27$

b) Adrienne deposits \$2000 in her savings account. The interest rate offered by the bank for this account is 8% compound interest per year. Calculate the number of years Adrienne needs to leave the money in her account for it to double in value.

An interest rate of 8% implies a common ratio of 1.08

The amount each year can be found using the term-to-term rule $u_{n+1} = 1.08 \times u_n$

$u_1 = 2000 \times 1.08 = 2160$

$u_2 = 2160 \times 1.08 = 2332.80$

$u_3 = 2332.80 \times 1.08 = 2519.42$

…

$u_9 = 3998.01$

$u_{10} = 4317.85$

Adrienne needs to leave the money in the account for 10 years in order for it to double in value.

Exercise 2.44

1. Identify which of the following are geometric sequences and which are not.

a) 2, 6, 18, 54 b) 25, 5, 1, $\frac{1}{5}$ c) 1, 4, 9, 16,

d) –3, 9, –27, 81 e) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ f) $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{8}{16}$

2. For the sequences in Q.1 that are geometric, calculate:

i) the common ratio r

ii) the next two terms

iii) a formula for the nth term.

3. The nth term of a geometric sequence is given by the formula $u_n = -6 \times 2^{n-1}$.

a) Calculate u_1, u_2 and u_3.

b) What is the value of n, if $u_n = -768$?

4. Part of a geometric sequence is given below:

…, –1, …, …, 64, … where $u_2 = -1$ and $u_5 = 64$. Calculate:

a) the common ratio r

b) the value of u_1

c) the value of u_{10}.

5. A homebuyer takes out a loan with a mortgage company for €200 000. The interest rate is 6% per year. If she is unable to repay any of the loan during the first 3 years, calculate the extra amount she will have to pay by the end of the third year, due to interest.

6. A car is bought for \$10 000. It loses value at a rate of 20% each year.
   a) Explain why the car is not worthless after 5 years.
   b) Calculate its value after 5 years.
   c) Explain why a depreciation of 20% per year means, in theory, that the car will never be worthless.

SECTION 11

Direct and inverse variation

Direct variation

Consider the tables below:

x	0	1	2	3	5	10
y	0	2	4	6	10	20

$y = 2x$

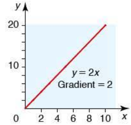
x	0	1	2	3	5	10
y	0	3	6	9	15	30

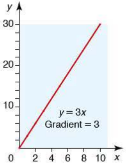
$y = 3x$

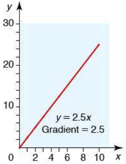
x	0	1	2	3	5	10
y	0	2.5	5	7.5	12.5	25

$y = 2.5x$

In each case y is directly proportional to x. This is written $y \propto x$. If any of these three tables is shown on a graph, the graph will be a straight line passing through the origin.







For any statement where $y \propto x$,

$y = kx$

where k is a constant equal to the gradient of the graph and is called the **constant of proportionality** or constant of variation.

Consider the tables below:

x	1	2	3	4	5
y	2	4	18	32	50

$y = 2x^2$

x	1	2	3	4	5
y	$\frac{1}{2}$	4	$13\frac{1}{2}$	32	$62\frac{1}{2}$

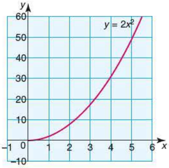
$y = \frac{1}{2}x^3$

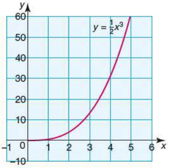
x	1	2	3	4	5
y	1	$\sqrt{2}$	$\sqrt{3}$	2	5

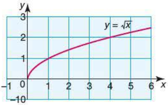
$y = \sqrt{x} = x^{\frac{1}{2}}$

In the cases above, y is directly proportional to x^n, where $n > 0$. This can be written as $y \propto x^n$.

The graphs of each of the three equations are shown below:

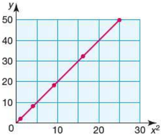






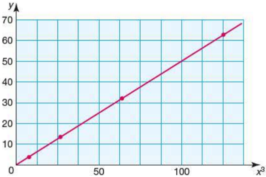
The graphs above, with (x, y) plotted, are not linear. However if the graph of $y = 2x^2$ is plotted as (x^2, y), then the graph is linear and passes through the origin demonstrating that $y \propto x^2$ as shown:

x	1	2	3	4	5
x^2	1	4	9	16	25
y	2	4	18	32	50



Similarly, the graph of $y = \frac{1}{2}x^3$ is curved when plotted as (x, y), but is linear and passes through the origin if it is plotted as (x^3, y) as shown:

x	1	2	3	4	5
x^3	1	8	27	64	125
y	$\frac{1}{2}$	4	$13\frac{1}{2}$	32	$62\frac{1}{2}$



The graph of $y = \sqrt{x}$ is also linear if plotted as $(\sqrt{x}, y)$.

Inverse variation

If y is inversely proportional to x, then $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

If a graph of y against $\frac{1}{x}$ is plotted, this too will be a straight line passing through the origin.

Worked examples

a) $y \propto x$. If $y = 7$ when $x = 2$, find y when $x = 5$.

$y = kx$
$7 = k \times 2$
$k = 3.5$

When $x = 5$,
$y = 3.5 \times 5$
$= 17.5$

b) $y \propto \frac{1}{x}$. If $y = 5$ when $x = 3$, find y when $x = 30$.

$y = \frac{k}{x}$

$5 = \frac{k}{3}$

$k = 15$

When $x = 30$,
$y = \frac{15}{30}$
$= 0.5$

Exercise 2.45

1. y is directly proportional to x. If $y = 6$ when $x = 2$, find:
a) the constant of proportionality
b) the value of y when $x = 7$
c) the value of y when $x = 9$
d) the value of x when $y = 9$
e) the value of x when $y = 30$.

2. y is directly proportional to x^2. If $y = 18$ when $x = 6$, find:
a) the constant of proportionality
b) the value of y when $x = 4$
c) the value of y when $x = 7$
d) the value of x when $y = 32$
e) the value of x when $y = 128$.

3. y is inversely proportional to x^3. If $y = 3$ when $x = 2$, find:
a) the constant of proportionality
b) the value of y when $x = 4$
c) the value of y when $x = 6$
d) the value of x when $y = 24$.

4. y is inversely proportional to x^2. If $y = 1$ when $x = 0.5$, find:
a) the constant of proportionality
b) the value of y when $x = 0.1$
c) the value of y when $x = 0.25$
d) the value of x when $y = 64$.

Exercise 2.46

1. Write each of the following in the form:

i) $y \propto x$
ii) $y = kx$.

a) y is directly proportional to x^3
b) y is inversely proportional to x^3
c) t is directly proportional to P
d) s is inversely proportional to t
e) A is directly proportional to r^2
f) T is inversely proportional to the square root of g

2. If $y \propto x$ and $y = 6$ when $x = 2$, find y when $x = 3.5$.

3. If $y \propto \frac{1}{x}$ and $y = 4$ when $x = 2.5$ find:

a) y when $x = 20$
b) x when $y = 5$.

4. If $p \propto r^2$ and $p = 2$ when $r = 2$, find p when $r = 8$.

5. If $m \propto \frac{1}{r^3}$ and $m = 1$ when $r = 2$, find:

a) m when $r = 4$
b) r when $m = 125$.

6. If $y \propto x^2$ and $y = 12$ when $x = 2$, find y when $x = 5$.

Exercise 2.47

1. If a stone is dropped off the edge of a cliff, the height (h metres) of the cliff is proportional to the square of the time (t seconds) taken for the stone to reach the ground.

A stone takes 5 seconds to reach the ground when dropped off a cliff 125 m high.

a) Write down a relationship between h and t, using k as the constant of variation.
b) Calculate the constant of variation.
c) Find the height of a cliff if a stone takes 3 seconds to reach the ground.
d) Find the time taken for a stone to fall from a cliff 180 m high.

2. The velocity (v metres per second) of a body is known to be proportional to the square root of its kinetic energy (e joules). When the velocity of a body is 120 m/s, its kinetic energy is 1600 J.

a) Write down a relationship between v and e, using k as the constant of variation.
b) Calculate the value of k.
c) If $v = 21$, calculate the kinetic energy of the body in joules.

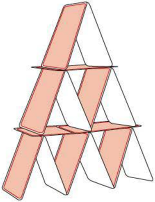
3. The length (l cm) of an edge of a cube is proportional to the cube root of its mass (m grams). It is known that if $l = 15$, then $m = 125$. Let k be the constant of variation.
   a) Write down the relationship between l, m and k.
   b) Calculate the value of k.
   c) Calculate the value of l when $m = 8$.
4. The power (P) generated in an electrical circuit is proportional to the square of the current (I amps). When the power is 108 watts, the current is 6 amps.
   a) Write down a relationship between P, I and the constant of variation, k.
   b) Calculate the value of I when $P = 75$ watts.

SECTION 12

Investigations, modelling and ICT

House of cards

The drawing shows a house of cards 3 layers high. 15 cards are needed to construct it.

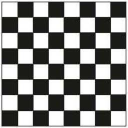


1. How many cards are needed to construct a house 10 layers high?
2. The world record is for a house 61 layers high. How many cards are needed to construct this house of cards?
3. Show that the general formula for a house n layers high requiring c cards is:

$$c = \tfrac{1}{2}n(3n + 1)$$

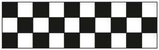
Chequered boards

A chessboard is an 8×8 square grid consisting of alternating black and white squares as shown:

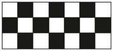


There are 64 unit squares of which 32 are black and 32 are white.

Consider boards of different sizes. The examples below show rectangular boards, each consisting of alternating black and white unit squares.



Total number of unit squares is 30
Number of black squares is 15
Number of white squares is 15

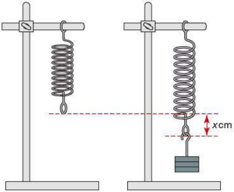


Total number of unit squares is 21
Number of black squares is 10
Number of white squares is 11

1. Investigate the number of black and white unit squares on different rectangular boards. Note: For consistency you made find it helpful to always keep the bottom right-hand square the same colour.
2. What is the number of black and white squares on a board $m \times n$ units?

Modelling: Stretching a spring

A spring is attached to a clamp stand as shown below.



Different weights are attached to the end of the spring , the mass (m) in grams is noted as is the amount by which the spring stretches (x) cm as shown on the right.

The data collected is shown in the table below:

Mass (g)	50	100	150	200	250	300	350	400	450	500
Extension (cm)	3.1	6.3	9.5	12.8	15.4	18.9	21.7	25.0	28.2	31.2

1. Plot a graph of mass against extension.
2. Describe the approximate relationship between the mass and the extension.
3. Draw a line of best fit through the data.
4. Calculate the equation of the line of best fit.
5. Use your equation to predict what the length of the spring would be for a weight of 275 g.
6. Explain why it is unlikely that the equation would be useful to find the extension if a weight of 5 kg was added to the spring.

Modelling: Linear programming

For each of the problems below, draw both axes numbered from 0 to 12. For each question:

a) write an inequality for each statement
b) graph the inequalities leaving the region which satisfies the inequalities unshaded
c) use your graph to state one solution which satisfies all the inequalities simultaneously.

1. A taxi firm has one car and one minibus for hire. During one morning it makes x car trips and y minibus trips.
   - It makes at least five car trips.
   - It makes between two and eight minibus trips.
   - The total number of car and minibus trips does not exceed 12.
2. A woman is baking bread and cakes. She makes p loaves and q cakes. She bakes at least five loaves and at least two cakes but no more than ten loaves and cakes in total.
3. A couple are buying curtains for their house. They buy m long curtains and n short curtains. They buy at least two long curtains. They also buy at least twice as many short curtains as long curtains. A maximum of 11 curtains are bought in total.
4. A shop sells large and small oranges. A girl buys L large oranges and S small oranges. She buys at least three but fewer than nine large oranges. She also buys fewer than six small oranges. The maximum number of oranges she can buy is 10.

ICT Activity 1

For each question, use a graphing package to plot the inequalities on the same pair of axes. Leave unshaded the region which satisfies all of them simultaneously.

1.	$y \leqslant x$	$y > 0$	$x \leqslant 3$
2.	$x + y > 3$	$y \leqslant 4$	$y - x > 2$
3.	$2y + x \leqslant 5$	$y - 3x - 6 < 0$	$2y - x > 3$

ICT Activity 2

You have seen that it is possible to solve some exponential equations by applying the laws of indices.

Use a graphics calculator and appropriate graphs to solve the following exponential equations:

1. $4^x = 40$
2. $3^x = 17$
3. $5^{x-1} = 6$
4. $3^{-x} = 0.5$

SECTION 13

Student assessments

Student assessment 1

1. Expand the following and simplify where possible:

a) $3(2x - 3y + 5z)$ b) $4p(2m - 7)$

c) $-4m(2mn - n^2)$ d) $4p^2(5pq - 2q^2 - 2p)$

e) $4x - 2(3x + 1)$ f) $4x(3x - 2) + 2(5x^2 - 3x)$

g) $\frac{1}{5}(15x - 10) - \frac{1}{3}(9x - 12)$ h) $\frac{x}{2}(4x - 6) + \frac{x}{4}(2x + 8)$

2. Factorise the following:

a) $16p - 8q$ b) $p^2 - 6pq$

c) $5p^2q - 10pq^2$ d) $9pq - 6p^2q + 12q^2p$

3. If $a = 4$, $b = 3$ and $c = -2$, evaluate the following:

a) $3a - 2b + 3c$ b) $5a - 3b^2$

c) $a^2 + b^2 + c^2$ d) $(a + b)(a - b)$

e) $a^2 - b^2$ f) $b^3 - c^3$

4. Rearrange the following formulae to make the **red** letter the subject:

a) $p = 4m + n$ b) $4x - 3y = 5z$

c) $2x = \frac{3y}{5p}$ d) $m(x + y) = 3w$

e) $\frac{pq}{4r} = \frac{mn}{t}$ f) $\frac{p + q}{r} = m - n$

Student assessment 2

1. Factorise the following fully:

a) $pq - 3rq + pr - 3r^2$ b) $1 - t^4$

c) $875^2 - 125^2$ d) $7.5^2 - 2.5^2$

2. Expand the following and simplify where possible:

a) $(x - 4)(x + 2)$ b) $(x - 8)^2$

c) $(x + y)^2$ d) $(x - 11)(x + 11)$

e) $(3x - 2)(2x - 3)$ f) $(5 - 3x)^2$

3. Factorise the following:

a) $x^2 - 4x - 77$ b) $x^2 - 6x + 9$

c) $x^2 - 144$ d) $3x^2 + 3x - 18$

e) $2x^2 + 5x - 12$ f) $4x^2 - 20x + 25$

4. Make the letter in **red** the subject of the formula:

a) $mf^2 = p$ b) $m = 5t^2$

c) $A = \pi r \quad p + q$ d) $\frac{1}{x} + \frac{1}{y} = \frac{1}{t}$

5. Simplify the following algebraic fractions:

a) $\frac{x^7}{x^3}$ b) $\frac{mn}{p} \times \frac{pq}{m}$

c) $\frac{(y^3)^3}{(y^2)^3}$ d) $\frac{28pq^2}{7pq^3}$

6. Simplify the following algebraic fractions:

a) $\frac{m}{11} + \frac{3m}{11} - \frac{2m}{11}$ b) $\frac{3p}{8} - \frac{9p}{16}$

c) $\frac{4x}{3y} - \frac{7x}{12y}$ d) $\frac{3m}{15p} + \frac{4n}{5p} - \frac{11n}{30p}$

7. Simplify the following:

a) $\frac{p}{5} + \frac{p}{4}$ b) $\frac{3m}{5} - \frac{2m}{4}$ c) $\frac{2p}{3} - \frac{3p}{4}$

8. Simplify the following:

a) $\frac{4}{(x-5)} + \frac{3}{(x-2)}$

b) $\frac{a^2 - b^2}{(a+b)^2}$

c) $\frac{x-2}{x^2 + x - 6}$

Student assessment 3

1. The volume of a cylinder is given by the formula $V = \pi r^2 h$, where h is the height of the cylinder and r is the radius.

a) Find the volume of a cylindrical post of length 7.5 m and a diameter of 30 cm.

b) Make r the subject of the formula.

c) A cylinder of height 75 cm has a volume of 6000 cm^3. Find its radius correct to three significant figures.

2. The formula $C = \frac{5}{9}(F - 32)$ can be used to convert temperatures in degrees Fahrenheit (°F) into degrees Celsius (°C).

a) What temperature in °C is equivalent to 150 °F?

b) What temperature in °C is equivalent to 12 °F?

c) Make F the subject of the formula.

d) Use your rearranged formula to find what temperature in °F is equivalent to 160 °C.

3. The height of Mount Kilimanjaro is given as 5900 m. The formula for the time taken, T hours, to climb to a height H metres is:

$$T = \frac{H}{1200} + k$$

where k is a constant.

a) Calculate the time taken, to the nearest hour, to climb to the top of the mountain if $k = 9.8$.
b) Make H the subject of the formula.
c) How far up the mountain, to the nearest 100 m, could you expect to be after 14 hours?

4. The formula for the volume V of a sphere is given as $V = \frac{4}{3}\pi r^3$.
a) Find V when $r = 5$ cm.
b) Make r the subject of the formula.
c) Find the radius of a sphere of volume 2500 m^3.

5. The cost £x of printing n newspapers is given by the formula $x = 1.50 + 0.05n$.
a) Calculate the cost of printing 5000 newspapers.
b) Make n the subject of the formula.
c) How many newspapers can be printed for £25?

Student assessment 4

For Q1.–4, solve the equations.

1. a) $y + 9 = 3$ b) $3x - 5 = 13$
c) $12 - 5p = -8$ d) $2.5y + 1.5 = 7.5$

2. a) $5 - p = 4 + p$ b) $8m - 9 = 5m + 3$
c) $11p - 4 = 9p + 15$ d) $27 - 5r = r - 3$

3. a) $\frac{p}{-2} = -3$ b) $6 = \frac{2}{5}x$
c) $\frac{m - 7}{5} = 3$ d) $\frac{4t - 3}{3} = 7$

4. a) $\frac{2}{5}(t - 1) = 3$ b) $5(3 - m) = 4(m - 6)$
c) $5 = \frac{2}{3}(x - 1)$ d) $\frac{4}{5}(t - 2) = \frac{1}{4}(2t + 8)$

5. Solve the following simultaneous equations:
a) $x + y = 11$ b) $5p - 3q = -1$
$x - y = 3$ $-2p - 3q = -8$
c) $3x + 5y = 26$ d) $2m - 3n = -9$
$x - y = 6$ $3m + 2n = 19$

Student assessment 5

1. The angles of a quadrilateral are x, $3x$, $(2x - 40)$ and $(3x - 50)$ degrees.
a) Construct an equation in terms of x.
b) Solve the equation.
c) Calculate the size of the four angles.

2. Three is subtracted from seven times a number. The result is multiplied by 5. If the answer is 55, calculate the value of the number by constructing an equation and solving it.

3. The interior angles of a pentagon are $9x$, $5x + 10$, $6x + 5$, $8x - 25$ and $10x - 20$ degrees. If the sum of the interior angles of a pentagon is 540°, find the size of each of the angles.

4. Solve $x^2 - x = 20$ by factorisation:

5. Solve $2x^2 - 7 = 3x$ by using the quadratic formula:

6. Solve the inequality $6 < 2x \leqslant 10$ and illustrate your answer on a number line.

7. For what values of m is $\frac{1}{m^2} > 0$ true?

Student assessment 6

1. The angles of a triangle are $x°$, $y°$ and 40°. The difference between the two unknown angles is 30°.
   a) Write down two equations from the information given above.
   b) What is the size of the two unknown angles?

2. The interior angles of a pentagon increase by 10° as you progress clockwise.
   a) Illustrate this information in a diagram.
   b) Write an expression for the sum of the interior angles.
   c) The sum of the interior angles of a pentagon is 540°. Use this to calculate the largest **exterior** angle of the pentagon.
   d) Illustrate, on your diagram, the size of each of the five exterior angles.
   e) Show that the sum of the exterior angles is 360°.

3. A flat sheet of card measures 12 cm by 10 cm. It is made into an open box by cutting a square of side x cm from each corner and then folding up the sides.
   a) Illustrate the box and its dimensions on a simple 3D sketch.
   b) Write an expression for the surface area of the outside of the box.
   c) If the surface area is 56 cm^2, form and solve a quadratic equation to find the value of x.

4. a) Show that $x - 2 = \frac{4}{x-3}$ can be written as $x^2 - 5x + 2 = 0$.
   b) Use the quadratic formula to solve $x - 2 = \frac{4}{x-3}$.

5. A right-angled triangle ABC has side lengths as follows: AB = x cm, AC is 2 cm shorter than AB, and BC is 2 cm shorter than AC.
   a) Illustrate this information on a diagram.
   b) Using this information, show that $x^2 - 12x + 20 = 0$.

c) Solve the above quadratic and hence find the length of each of the three sides of the triangle.

6. Solve the following inequalities:

a) $5 + 6x \leqslant 47$ b) $4 \geqslant \frac{y+3}{3}$

7. Find the range of values for which:

a) $3 \leqslant 3p < 12$ b) $24 < 8(x - 1) \leqslant 48$

Student assessment 7

1. Using indices, simplify the following:

a) $3 \times 2 \times 2 \times 3 \times 27$

b) $2 \times 2 \times 4 \times 4 \times 4 \times 2 \times 32$

2. Write the following out in full:

a) 6^5 b) 2^{-5}

3. Work out the value of the following without using a calculator:

a) $3^3 \times 10^3$ b) $1^{-4} \times 5^3$

4. Simplify the following using indices:

a) $2^4 \times 2^3$ b) $7^5 \times 7^2 \times 3^4 \times 3^8$

c) $\frac{4^8}{2^{10}}$ d) $\frac{(3^3)^4}{27^3}$

e) $\frac{7^6 \times 4^2}{4^3 \times 7^6}$ f) $\frac{8^{-2} \times 2^6}{2^{-2}}$

5. Without using a calculator, evaluate the following:

a) $5^2 \times 5^{-1}$ b) $\frac{4^5}{4^3}$

c) $\frac{7^{-5}}{7^{-7}}$ d) $\frac{3^{-5} \times 4^2}{3^{-6}}$

6. Find the value of x in each of the following:

a) $2^{(2x+2)} = 128$ b) $\frac{1}{4^{-x}} = \frac{1}{2}$

c) $3^{(-x+4)} = 81$ d) $8^{-3x} = \frac{1}{4}$

Student assessment 8

1. Evaluate the following without the use of a calculator:

a) $64^{\frac{1}{6}}$ b) $27^{\frac{4}{3}}$

c) $9^{-\frac{1}{2}}$ d) $512^{\frac{2}{3}}$

e) $\sqrt[3]{27}$ f) $\sqrt[4]{16}$

g) $\frac{1}{36^{-\frac{1}{2}}}$ h) $\frac{2}{64^{-\frac{2}{3}}}$

2. Evaluate the following without the use of a calculator:

a) $\dfrac{25^{\frac{1}{2}}}{9^{-\frac{1}{2}}}$　　b) $\dfrac{4^{\frac{5}{2}}}{2^3}$

c) $\dfrac{27^{\frac{4}{3}}}{3^3}$　　d) $25^{\frac{3}{2}} \times 5^2$

e) $4^{\frac{6}{4}} \times 4^{-\frac{1}{2}}$　　f) $\dfrac{27^{\frac{2}{3}} \times 3^{-3}}{9^{-\frac{1}{2}}}$

g) $\dfrac{(4^2)^{-\frac{1}{4}} \times 9^{\frac{3}{2}}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}}$　　h) $\dfrac{(5^{\frac{1}{3}})^{\frac{1}{2}} \times 5^{\frac{5}{6}}}{4^{-\frac{1}{2}}}$

3. Draw a pair of axes with x from –4 to 4 and y from 0 to 18.

a) Plot a graph of $y = 4^{-\frac{x}{2}}$.
b) Use your graph to estimate when $4^{-\frac{x}{2}} = 6$.

Student assessment 9

1. For each of the following arithmetic sequences:
i) write down a formula for the nth term
ii) calculate the 10th term.
a) 1, 5, 9, 13, …　　b) 1, –2, –5, –8, …

2. For both of the following, calculate u_5 and u_{100}:

a) $u_n = 6n - 3$　　b) $u_n = -\frac{1}{2}n + 4$

3. Copy and complete both of the following tables of arithmetic sequences:

a)

Position	1	2	3	10		*n*
Term	17	14			–55	

b)

Position	2	6	10		*n*
Term	–4	–2		35	

4. A girl deposits \$300 in a bank account. The bank offers 7% interest per year.
Assuming the girl does not take any money out of the account calculate:
a) the amount of money in the account after 8 years
b) the minimum number of years the money must be left in the account, for the amount to be greater than \$350.

5. A computer loses 35% of its value each year. If the computer cost €600 new, calculate:
a) its value after 2 years
b) its value after 10 years.

6. Part of a geometric sequence is given below:

   ...., ..., 27, ..., ..., –1

   where $u_3 = 27$ and $u_6 = -1$.
   Calculate:
   a) the common ratio r
   b) the value u_1
   c) the value of n if $u_n = -\frac{1}{81}$.

7. Using a table of differences if necessary, calculate the rule for the nth term of the sequence 8, 24, 58, 116, 204, … .

8. Using a table of differences, calculate the rule for the nth term of the sequence 10, 23, 50, 97, 170, … .

Student assessment 10

1. $y = kx$. When $y = 12, x = 8$.
   a) Calculate the value of k.
   b) Calculate y when $x = 10$.
   c) Calculate y when $x = 2$.
   d) Calculate x when $y = 18$.

2. $y = \frac{k}{x}$. When $y = 2, x = 5$.
   a) Calculate the value of k.
   b) Calculate y when $x = 4$.
   c) Calculate x when $y = 10$.
   d) Calculate x when $y = 0.5$.

3. $p = kq^3$. When $p = 9, q = 3$.
   a) Calculate the value of k.
   b) Calculate p when $q = 6$.
   c) Calculate p when $q = 1$.
   d) Calculate q when $p = 576$.

4. $m = \frac{k}{\sqrt{n}}$. When $m = 1, n = 25$.
   a) Calculate the value of k.
   b) Calculate m when $n = 16$.
   c) Calculate m when $n = 100$.
   d) Calculate n when $m = 5$.

5. $y = \frac{k}{x^2}$. When $y = 3, x = \frac{1}{3}$.
   a) Calculate the value of k.
   b) Calculate y when $x = 0.5$.
   c) Calculate both values of x when $y = \frac{1}{12}$.
   d) Calculate both values of x when $y = \frac{1}{3}$.

Student assessment 11

1. y is inversely proportional to x.

a) Copy and complete the table below:

x	1	2	4	8	16	32
y				4		

b) What is the value of x when $y = 20$?

2. Copy and complete the tables below:

a) $y \propto x$

x	1	2	4	5	10
y		10			

b) $y \propto \frac{1}{x}$

x	1	2	4	5	10
y	20				

c) $y \propto \sqrt{x}$

x	4	16	25	36	64
y	4				

3. The pressure (P) of a given mass of gas is inversely proportional to its volume (V) at a constant temperature. If $P = 4$ when $V = 6$, calculate:

a) P when $V = 30$

b) V when $P = 30$.

4. The gravitational force (F) between two masses is inversely proportional to the square of the distance (d) between them. If $F = 4$ when $d = 5$, calculate:

a) F when $d = 8$

b) d when $F = 25$.

TOPIC

3 Functions

This topic will cover the following syllabus content:

3.1 Notation; Domain and range; Mapping diagrams

3.2 Recognition of the following function types from the shape of their graphs:

linear $f(x) = ax + b$
quadratic $f(x) = ax^2 + bx + c$
cubic $f(x) = ax^3 + bx^2 + cx + d$
reciprocal $f(x) = a/x$
exponential $f(x) = a^x$ with $0 < a < 1$ or $a > 1$
absolute value $f(x) = |ax + b|$
trigonometric $f(x) = a\sin(bx)$; $a\cos(bx)$; $\tan x$

3.3 Determination of at most two of a, b, c or d in simple cases of 3.2

3.4 Finding the quadratic function given: vertex and another point; x-intercepts and a point; vertex or x-intercepts with $a = 1$

3.5 Understanding of the concept of asymptotes and identification of examples

3.6 Use of a graphics calculator to: sketch the graph of a function; produce a table of values; find zeros, local maxima or minima; find the intersection of the graphs of functions.

3.7 Simplified formulae for expressions such as $f(g(x))$ where $g(x)$ is a linear expression

3.8 Description and identification, using the language of transformations, of the changes to the graph of $y = f(x)$ when
$y = f(x) + k$, $y = k\,f(x)$, $y = f(x + k)$

3.9 Inverse function f^{-1}

3.10 Logarithmic function as the inverse of the exponential function
$y = a^x$ equivalent to $x = \log_a y$
Rules for logarithms corresponding to rules for exponents
Solution to $a^x = b$ as $x = \log b/\log a$

Sections

SECTION 1 The Chinese

Chinese mathematicians were the first to discover various algebraic and geometric principles. The textbook *Nine Chapters on the Mathematical Art* has special importance. *Nine Chapters* (known in Chinese as *Jiu Zhang Suan Shu* or *Chiu Chang Suan Shu*) was probably written during the early Han Dynasty (about 165BC) by Chang Tshang.

Chang's book gives methods of arithmetic (including cube roots) and algebra (including a solution of simultaneous equations), uses the decimal system with zero and negative numbers, proves Pythagoras' theorem and includes a clever geometric proof that the perimeter of a right triangle multiplied by the radius of its inscribing circle equals the area of its circumscribing rectangle.

Chang was concerned with the ordinary lives of the people. He wrote 'For a civilization to endure and prosper, it must give its citizens order and fairness' so three chapters were concerned with ratio and proportion, so that 'rice and other cereals can be planted in the correct proportion to our needs, and the ratio of taxes could be paid fairly'.

Nine Chapters was probably based on earlier books but, even so, this book had great historical importance. It was the main Chinese mathematical text for centuries, and had great influence throughout the Far East. Some of the teachings made their way to India and from there to the Islamic world and Europe. The Hindus may have borrowed the decimal system itself from books like *Nine Chapters*.

In AD600 Wang Xiaotong wrote *The Continuation of Ancient Mathematics* which included work on squares, cubes and their roots.

SECTION 2 Function notation

Functions as a mapping

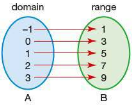
Consider the equation $y = 2x + 3$. It describes the relationship between two variables x and y. In this case, 3 is added to twice the value of x to produce y.

A function is a particular type of relationship between two variables. It has certain characteristics.

Consider the equation $y = 2x + 3$ for values of x within $-1 \leqslant x \leqslant 3$.

A table of results can be constructed and a mapping drawn.

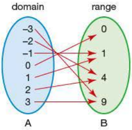
x	y
−1	1
0	3
1	5
2	7
3	9



With a function, each value in set B (the **range**) is produced from one value in set A (the **domain**). The relationship can be written as a function:

$$f(x) = 2x + 3; -1 \leqslant x \leqslant 3$$

or $$f : x \mapsto = 2x + 3; -1 \leqslant x \leqslant 3$$



It is also usual to include the domain after the function, as a different domain will produce a different range.

The mapping from A to B can be a one-to-one mapping or a many-to-one mapping.

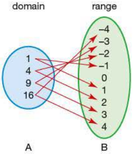
The function above, $f(x) = 2x + 3; -1 \leqslant x \leqslant 3$, is a one-to-one function as one value in the domain maps onto one value in the range. However the function $f(x) = x^2; x \in \mathbb{Z}$ is a many-to-one function, as a value in the range can be generated by more than one value in the domain as shown.

It is important to understand that one value in the domain (set A) maps to only one value in the range (set B). Therefore the mapping shown is the function $f(x) = x^2; x \in \mathbb{Z}$.

Some mappings will not represent functions, for example consider the relationship $y = \pm\sqrt{x}$.

The following table and mapping diagram can be produced:

x	y
1	±1
4	±2
9	±3
16	±4



This relationship is not a function as a value in the domain produces more than one value in the range.

It is also important to remember the mathematical notation used to define a domain. The principal ones are shown below:

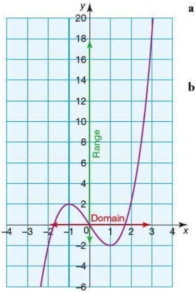
Notation	Meaning
$\mathbb{Z}$	The set of integers {0, ±1, ±2, ±3, …}
$\mathbb{Z}^+$	The set of positive integers {1, 2, 3, …}
$\mathbb{N}$	The set of natural numbers {0, 1, 2, 3, …} i.e. positive integers and zero
$\mathbb{Q}$	The set of rational numbers i.e. can be expressed as a fraction
$\mathbb{R}$	The set of real numbers i.e. numbers that exist

Note: If a domain is not specified then it is assumed to be all real values $\mathbb{R}$.

Calculating the range from the domain

The domain is the set of input values and the range is the set of output values for a function. (Note that the range is not the difference between the greatest and least values as in statistics.) The range is therefore not only dependent on the function itself, but also on the domain.

Worked examples



Calculate the range for the following functions:

a) $f(x) \mapsto x^3 - 3x; -2 \leqslant x \leqslant 3$

The graph of the function is shown opposite. As the domain is restricted to $-2 \leqslant x \leqslant 3$, the range is limited from –2 to 18.

This is written as: Range $-2 \leqslant f(x) \leqslant 18$.

b) $f(x) \mapsto x^3 - 3x; x \in \mathbb{R}$

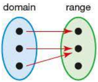
The graph will be similar to the one to the left except that the domain is not restricted. As the domain is for all real values of x, this implies that any real number can be an input value. As a result, the range will also be all real values.

This is written as: Range $f(x) \in \mathbb{R}$.

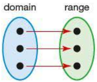
Exercise 3.1

1. Which of the following mappings shows a function?

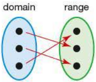
a)



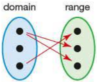
b)



c)



d)



Give the domain and range of each of the functions in Q.2–8.

2. $f(x) = 2x - 1; -1 \leqslant x \leqslant 3$

3. $f(x) = 3x + 2; -4 \leqslant x \leqslant 0$

4. $f(x) = -x + 4; -4 \leqslant x \leqslant 4$

5. $f(x) = x^2 + 2; -3 \leqslant x \leqslant 3$

6. $f(x) = x^2 + 2; x \in \mathbb{R}$

7. $f(x) = -x^2 + 2; 0 \leqslant x \leqslant 4$

8. $f(x) = x^3 - 2; -3 \leqslant x \leqslant 1$

SECTION 3

Recognising graphs of common functions

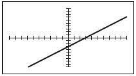
Graphs of functions take many different forms. It is important to be able to identify common functions and their graphs.

The linear function

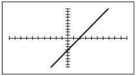
A linear function takes the form $f(x) = ax + b$ and when graphed produces a straight line.

Three different linear functions are shown below:

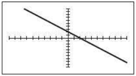
$f(x) = x - 3$



$f(x) = 2x - 4$



$f(x) = -x + 2$

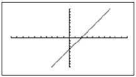
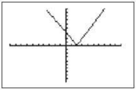


The values of a and b affect the orientation and position of the line.

For the function $f(x) = x - 3$, $a = 1$ and $b = -3$.
For the function $f(x) = 2x - 4$, $a = 2$ and $b = -4$.
For the function $f(x) = -x + 2$, $a = -1$ and $b = 2$.

You can use your graphics calculator to investigate linear functions and determine the effects that different values of a and b have on the graph.

The instructions below will remind you how to graph the function $f(x) = 2x - 4$ using your graphics calculator:

Casio	
MENU 5 to select the graphing menu. 2 X,θ,T − 4 EXE to enter the function. F6 to graph the function.	
Texas	
Y= to select the function. 2 X,T,θ,n − 4 ENTER to enter the function. GRAPH to graph the function.	

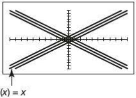
Exercise 3.2

1. Use your graphics calculator to investigate the effect of b on the orientation or position of functions of the type $f(x) = ax + b$.
   a) By keeping the value of a constant and changing the value of b, write down five different linear functions.
   b) Using your graphics calculator, graph each of the five functions.
   c) Sketch your functions, labelling each clearly.
   d) Write a short conclusion about the effect of b on the graph.

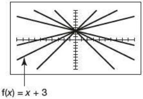
2. Use your graphics calculator to investigate the effect of a on the orientation or position of functions of the type $f(x) = ax + b$.
   a) By keeping the value of b constant and changing the value of a, write down five different linear functions.
   b) Using your graphics calculator, graph each of the five functions.
   c) Sketch your functions, labelling each clearly.
   d) Write a short conclusion about the effect of a on the graph.

3. Use your graphics calculator to produce a **similar** screen to those shown below. The equation of one of the functions is given each time.

a)



b)

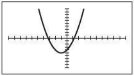


The quadratic function

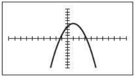
A quadratic function takes the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. The graph of a quadratic function also has a characteristic shape that you can use to identify that a function is quadratic.

Two quadratic functions are shown below:

$f(x) = x^2 + 2x - 4$



$f(x) = -x^2 + 2x + 4$



The graphs of quadratic functions are always either U-shaped or upside down U-shaped. The values of a, b and c affect the shape and position of the graph.

Exercise 3.3

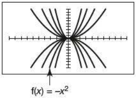
1. Use your graphics calculator to investigate the effect of c on the shape or position of functions of the type $f(x) = ax^2 + bx + c$.
   - a) By keeping the values of a and b constant and changing the value of c, write down five different quadratic functions.
   - b) Using your graphics calculators, graph each of the five functions.
   - c) Sketch your functions, labelling each clearly.
   - d) Write a short conclusion about the effect of c on the graph.

2. Use your graphics calculator to investigate the effect of a on the shape or position of functions of the type $f(x) = ax^2 + bx + c$.
   - a) By keeping the values of b and c constant and changing the value of a, write down five different quadratic functions
   - b) Using your graphics calculator, graph each of the five functions.
   - c) Sketch your functions, labelling each clearly.
   - d) Write a short conclusion about the effect of a on the graph.

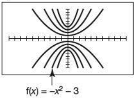
3. Use your graphics calculator to investigate the effect of b on the shape or position of functions of the type $f(x) = ax^2 + bx + c$.
   - a) By keeping the values of a and c constant and changing the value of b, write down five different quadratic functions.
   - b) Using your graphics calculator, graph each of the five functions.
   - c) Sketch your functions, labelling each clearly.
   - d) Write a short conclusion about the effect of b on the graph.

4. Use your graphics calculator to produce a **similar** screen to those shown below. The equation of one of the functions is given each time.

a)



b)



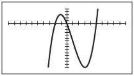
The cubic function

A cubic function takes the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$.

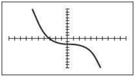
They also have an identifiable shape.

Two examples are shown below:

$f(x) = x^3 - 3x^2 - 10x$
where $a = 1, b = -3, c = -10$
and $d = 0$



$f(x) = -x^3 - 2$
where $a = -1, b = 0, c = 0$
and $d = -2$



The shape of a cubic function has a characteristic 'S' shape. It can be tighter as in the example on the left, or more stretched as shown in the example on the right.

Exercise 3.4

1. a) Use your graphics calculator to investigate the effect of a on the shape or position of functions of the type $f(x) = ax^3 + bx^2 + cx + d$.

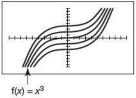
b) Write a short conclusion about the effect of a on the graph.

2. a) Use your graphics calculator to investigate the effect of d on the shape or position of functions of the type $f(x) = ax^3 + bx^2 + cx + d$.

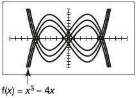
b) Write a short conclusion about the effect of d on the graph.

3. Use your graphics calculator to produce a **similar** screen to those shown below. The equation of one of the functions is given each time.

a)



b)



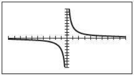
The reciprocal function

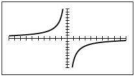
You will have encountered the term reciprocal before, for example the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}}$ which simplifies to $\frac{3}{2}$. The reciprocal of x is therefore $\frac{1}{x}$.

Functions where x appears in the denominator are reciprocal functions and take the form $f(x) = \frac{a}{x}$, where $a \neq 0$.

The graphs of reciprocal functions have particular characteristics as shown in the two examples below:

$f(x) = \frac{1}{x}$ $f(x) = -\frac{2}{x}$





In each case the graphs get closer to the axes but do not actually meet or cross them. This is because for the function $f(x) = \frac{a}{x}$, as $x \to \pm\infty$ then $\frac{a}{x} \to 0$ and as $x \to 0$ then $\frac{a}{x} \to \pm\infty$.

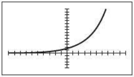
The axes are known as **asymptotes**. An asymptote is a line to which a curve gets closer and closer but never actually meets.

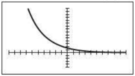
The exponential function

Until now all the functions you have encountered have a variable raised to a power, for example $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$ and $f(x) = \frac{1}{x} = x^{-1}$. With an exponential function, the variable is the power.

An exponential function will typically take the form $f(x) = a^x$ where $0 < a < 1$ or $a > 1$. The graphs of two exponential functions are shown below:

$f(x) = 2^x$ $f(x) = \left(\frac{1}{2}\right)^x$





The graphs show that the x-axis is an asymptote to each of the curves. This is because for the function $f(x) = 2^x$, as $x \to -\infty$ then $2^x \to 0$. This can be seen by applying the laws of indices where $2^{-x} = \frac{1}{2^x}$.

Similarly for the function $f(x) = \left(\frac{1}{2}\right)^x$, as $x \to \infty$ then $f(x) = \left(\frac{1}{2}\right)^x \to 0$.

Exercise 3.5

1. a) Use your graphics calculator to investigate the effect of *a* on the shape or position of functions of the type $f(x) = \frac{a}{x}$. Remember to include negative and positive value of *a*.

b) Write a short conclusion about the effect of *a* on the graph.

2. a) Use your graphics calculator to graph the following functions on the same screen:

$f(x) = 2^x$ $\quad f(x) = 3^x$ $\quad f(x) = 4^x$

b) Describe two characteristics that the graphs of all three functions share.

3. a) Use your graphics calculator to graph each of the following pairs of functions simultaneously:

i) $f(x) = 3^x$ and $f(x) = \left(\frac{1}{3}\right)^x$

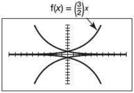
ii) $f(x) = 4^x$and $f(x) = \left(\frac{1}{4}\right)^x$

iii) $f(x) = \left(\frac{3}{2}\right)^x$ and $f(x) = \left(\frac{2}{3}\right)^x$

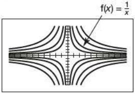
b) Comment on the relationship between each pair of graphs.

4. Use your graphics calculator to produce a **similar** screen to those shown below. The equation of one of the functions is given each time.

a)



b)



Absolute functions

The concept of an absolute value was introduced in Topic 1. The absolute value of a number refers to its magnitude rather than its sign, i.e. $|-3| = 3$.

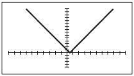
Similarly, absolute functions relate to their magnitude.

Consider the absolute function $f(x) = |2x - 1|$ when different values of *x* are substituted:

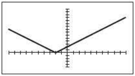
$f(3) = |2 \times 3 - 1| = |5| = 5$
$f(2) = |2 \times 2 - 1| = |3| = 3$
$f(1) = |2 \times 1 - 1| = |1| = 1$
$f(0) = |2 \times 0 - 1| = |-1| = 1$
$f(-1) = |(2 \times -1) - 1| = |-3| = 3$

Because the result of an absolute function is always positive, the graph of an absolute function will never go below the x-axis. The graphs of two absolute functions are shown below:

$f(x) = |2x - 1|$



$f(x) = |x + 2|$



Using the graphics calculator to graph absolute functions

A graphics calculator can be used to graph absolute functions. For example, to graph the function $f(x) = |x - 2|$, follow the steps below:

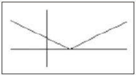
Casio

MENU 5 to select the graphing mode.

OPTN F5 F1 , to select 'Abs'.

(X,θ,T − 2) EXE to enter the absolute function.

F6 to graph the function.



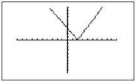
Texas

Y=

MATH › 1 to select 'abs'.

X,T,θ,n − 2) ENTER to enter the absolute function.

GRAPH to graph the function.



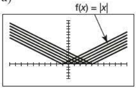
Note: It is important to enter the function in brackets so that the absolute value is calculated for the whole function, i.e. Abs(x – 2) calculates the absolute value of (x – 2), whilst Abs x – 2 only calculates the absolute value of x from which 2 is then subtracted. If this happens, a negative result is possible.

Exercise 3.6

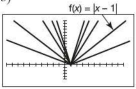
1. a) Graph each of the following pairs of functions on your graphics calculator:
    i) $f(x) = x$, and $f(x) = |x|$
    ii) $f(x) = x - 3$, and $f(x) = |x - 3|$
    iii) $f(x) = \frac{1}{2}x + 1$, and $f(x) = |\frac{1}{2}x + 1|$
    iv) $f(x) = -2x - 2$, and $f(x) = |-2x - 2|$
   b) Comment on the graphical relationship between a function and its corresponding absolute function.

2. Use your graphics calculator to produce a **similar** screen to those shown below. The equation of one of the functions is given each time.

a)



b)



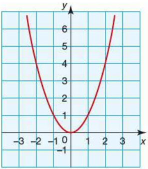
Trigonometric functions and their properties are dealt with in Topic 8.

SECTION 4 Transforming graphs

When a function undergoes a single transformation, the shape or position of its graph changes. This change in shape or position depends on the type of transformation. This section focuses on two transformations in particular:

1. A translation
2. A stretch

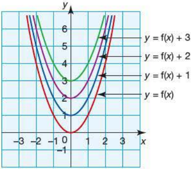
Let $f(x) = x^2$. The graph $y = f(x)$ is therefore the graph of $y = x^2$ as shown below:



The graph of the function $y = \text{f}(x) + 1$ is therefore the graph of $y = x^2 + 1$.

The graph of the function $y = \text{f}(x) + 2$ is the same as the graph of $y = x^2 + 2$ and the graph of the function $y = \text{f}(x) + 3$ is the same as the graph of $y = x^2 + 3$.

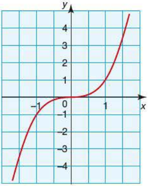
These four functions are plotted on the same axes as shown:



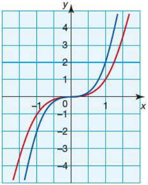
These graphs show that $y = \text{f}(x)$ is translated $\begin{pmatrix}0\\1\end{pmatrix}$ to map onto $y = \text{f}(x) + 1$. Similarly, $y = \text{f}(x)$ is translated $\begin{pmatrix}0\\2\end{pmatrix}$ to map onto $y = \text{f}(x) + 2$ and translated $\begin{pmatrix}0\\3\end{pmatrix}$ to map onto $y = \text{f}(x) + 3$.

Therefore $y = \text{f}(x) + k$ is a translation of $\begin{pmatrix}0\\k\end{pmatrix}$ of the function $y = \text{f}(x)$.

When $\text{f}(x) = x^3$, the graph of the function $y = \text{f}(x)$ is the same as the graph of $y = x^3$. This is drawn below:



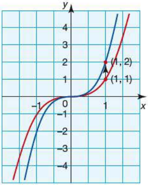
Sketching the functions $y = f(x)$ and $y = 2f(x)$ on the same axes produces the following graph:



It is not clear from the graphs what transformation has occurred. However, this can be found by looking at the coordinate of a point on the original function $y = f(x)$ and finding the point onto which it is mapped on the function $y = 2f(x)$.

i.e. The point $(1, 1)$ lies on the graph of $y = x^3$.
Keeping the x value as 1 and substituting it into the function $y = 2x^3$ gives a y value of 2.

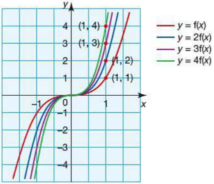
Therefore the point $(1, 1)$ has been mapped onto the point $(1, 2)$ as shown:



Similarly, the point $(2, 8)$ is mapped onto the point $(2, 16)$.

The effect of mapping $y = f(x)$ onto $y = 2f(x)$ is a stretch of scale factor 2 parallel to the y-axis.

Graphing the functions $y = \mathrm{f}(x)$, $y = 2\mathrm{f}(x)$, $y = 3\mathrm{f}(x)$ and $y = 4\mathrm{f}(x)$ on the same axes produces the graphs shown below:

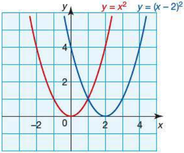


It can be seen from the point (1, 1) that the effect of mapping $y = \mathrm{f}(x)$ onto $y = 3\mathrm{f}(x)$ is a stretch of scale factor 3 parallel to the y-axis and the effect of mapping $y = \mathrm{f}(x)$ onto $y = 4\mathrm{f}(x)$ is a stretch of scale factor 4 parallel to the y-axis.

In general, therefore, mapping $y = \mathrm{f}(x)$ onto $y = k\mathrm{f}(x)$ is a stretch of scale factor k parallel to the y-axis.

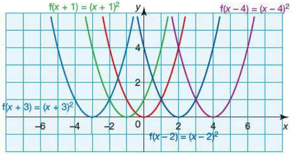
Earlier in this section the transformations of $y = \mathrm{f}(x) + k$ were investigated. The constant k acted externally to the original function $y = \mathrm{f}(x)$. If the constant is incorporated within the original function, a different transformation occurs.

Let $\mathrm{f}(x) = x^2$ be the original function. This is represented by the equation $y = x^2$. If x is substituted by $(x - 2)$, the function becomes $\mathrm{f}(x - 2) = (x - 2)^2$. This is represented by the equation $y = (x - 2)^2$. Graphing both on the same axes produces the graphs below:



$y = x^2$ is mapped onto $y = (x - 2)^2$ by the transformation $\begin{pmatrix}2\\0\end{pmatrix}$.

The graphs of the functions $f(x) = x^2$, $f(x - 2) = (x - 2)^2$, $f(x - 4) = (x - 4)^2$, $f(x + 1) = (x + 1)^2$ and $f(x + 3) = (x + 3)^2$ are shown on the same axes below:



The transformations are each horizontal translations.
In general, therefore, mapping $y = f(x)$ onto $y = f(x + k)$ is a translation of $\begin{pmatrix} -k \\ 0 \end{pmatrix}$.

Exercise 3.7

1. Sketch the graph of $y = x^2$. Use transformations to sketch, on the same axes, both of the following. Label each graph clearly.

a) $y = (x - 1)^2$ b) $y = x^2 - 3$

2. Sketch the graph of $y = x^3$. Use transformations to sketch, on the same axes, both of the following. Label each graph clearly.

a) $y = (x + 3)^3$ b) $y = \frac{1}{2}x^3$

3. a) Sketch the graph of $y = 2^x$. Use transformations to sketch, on the same axes, both of the following. Label each graph clearly.

i) $y = 2^x + 2$ ii) $y = 2^{(x - 2)}$

b) Give the equation of any asymptote in Q.3(a)(i).

c) Give the equation of any asymptote in Q.3(a)(ii).

4. a) Sketch the graph of $y = \frac{1}{x}$. Use transformations to sketch, on the same axes, both of the following. Label each graph clearly.

i) $y = \frac{1}{3x}$ ii) $y = \frac{1}{x + 2}$

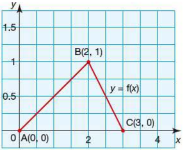
b) Give the equation of any asymptote in Q.4(a)(i).

c) Give the equation of any asymptote in Q.4(a)(ii).

5. Describe mathematically the transformation that maps the graph of $y = f(x)$ onto:

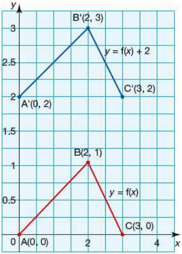
a) $y = f(x) - 2$ b) $y = 5f(x)$ c) $y = f(x - 4)$

Worked example The sketch shows the graph of $y = \mathrm{f}(x)$. Points A, B and C have coordinates as shown:



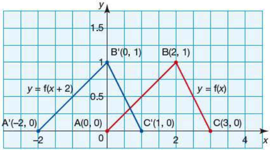
i) Sketch the graph of $y = \mathrm{f}(x) + 2$. Mark the images of A, B and C under the transformation and state their coordinates.

The transformation $y = \mathrm{f}(x) + 2$ is a translation of $\begin{pmatrix}0\\2\end{pmatrix}$. Each point on $y = \mathrm{f}(x)$ is therefore translated 2 units vertically upwards. The graph of $y = \mathrm{f}(x) + 2$ and the images of A, B and C, labelled A', B' and C', are therefore as shown:



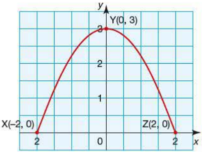
ii) Sketch the graph of $y = f(x + 2)$. Mark the images of A, B and C under the transformation and state their coordinates.

The transformation $y = f(x + 2)$ is a translation of $\begin{pmatrix}-2\\0\end{pmatrix}$. Each point on $y = f(x)$ is therefore translated 2 units horizontally to the left. The graph of $y = f(x + 2)$ and images of A, B and C, labelled A', B' and C', are therefore as shown:



Exercise 3.8

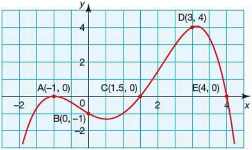
1. Sketch the graph of $y = f(x)$ shown below:



a) On the same axes, sketch the graph of $y = f(x - 3)$, stating clearly the coordinates of the images of the points X, Y and Z.

b) Describe the transformation that maps $y = f(x)$ onto $y = f(x - 3)$.

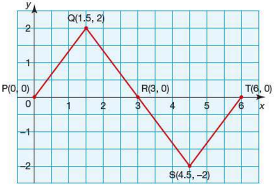
2. Sketch the graph of $y = g(x)$ shown below:



a) On the same axes, sketch the graph of $y = 3g(x)$, stating clearly the coordinates of the images of the points A, B, C, D and E.

b) Describe the transformation that maps $y = g(x)$ onto $y = 3g(x)$.

3. Sketch the graph of $y = f(x)$ shown below:



a) On the same axes, sketch the graph of $y = f(x) - 2$, stating clearly the coordinates of the images of the points P, Q, R, S and T.

b) Describe the transformation that maps $y = f(x)$ onto $y = f(x) - 2$.

4. a) Given that $f(x) = x^2$ sketch the graphs of each of the following functions on a separate pair of axes:

i) $y = f(x) + 4$ ii) $y = f(x + 2)$ iii) $y = \frac{1}{3}f(x)$

b) Write the equation of y in terms of x for each of the functions in Q.4(a).

5. a) Given that $f(x) = \frac{1}{x}$, sketch the graphs of each of the following functions on a separate pair of axes.

i) $y = 3f(x)$ ii) $y = f(x - 4)$ iii) $y = f(x) - 2$

b) Write the equation of y in terms of x for each of the functions in Q.5(a).

c) Write the equation of any asymptotes in the graphs of the functions.

SECTION 5

Using a graphics calculator to sketch and analyse functions

The graphics calculator, introduced in the introductory topic, is a powerful tool to help understand graphs of functions and their properties. This section recaps some of the features that are particularly useful for checking your answers to some questions in the latter part of this topic.

Worked example Consider the function $f(x) = -x^3 + 9x^2 - 24x + 16$.

i) Plot the function on a graphics calculator.

Casio

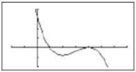
MENU 5 to select the graphing mode.

(−) X,θ,T ^ 3 + 9 X,θ,T

x^2 − 2 4 X,θ,T +

1 6 EXE to enter the function.

F6 to graph the function.



Note: It may be necessary to adjust the axes using the viewing window V-Window F3 so that the graph appears as above.

Texas

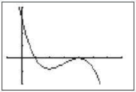
Y=

(−) X,T,θ,n ^ 3 + 9

X,T,θ,n x^2 − 2 4 X,T,θ,n

+ 1 6 ENTER to enter the function.

GRAPH to graph the function.



Note: It may be necessary to adjust the axes using the WINDOW button so that the graph appears as above.

ii) Find the roots of the function, i.e. where its graph intersects the x-axis.

Casio

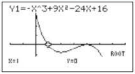
With the graph on the screen

to select the 'graph solve' menu.

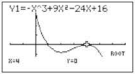
to find the 'roots' of the graph.

The calculator calculates the coordinate of the first root.

to scroll and find any other roots.



```
Y1=-X^3+9X²-24X+16
ROOT
X=1          Y=0
```



```
Y1=-X^3+9X²-24X+16
ROOT
X=4          Y=0
```

Texas

With the graph on the screen

to select the 'calc' menu.

to find where the function is zero.


to move the cursor to the left of the first root,

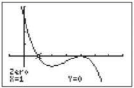

to select a left bound.


to move the cursor to the right of the root,

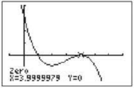

to select a right bound.

to calculate its coordinate.

Repeat the process to find the second root.



```
Zero
X=1          Y=0
```



```
Zero
X=3.9999979  Y=0
```

Note: Sometimes the solutions are only an approximation and not exact.

iii) Find where the graph intersects the y-axis.

This can be done easily without a graphics calculator, as the intersection with the y-axis occurs when $x = 0$. Substituting $x = 0$ into the equation gives the solution $y = 16$.

The graphics calculator can be used as follows:

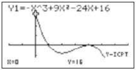
Casio

With the graph on the screen

G-Solv F5 to select 'graph solve' menu.

Sketch F4 to select 'Y-ICPT'.

The calculator gives the coordinate of the y-intercept.



Y1=-X^3+9X²-24X+16

Y-ICPT

X=0 Y=16

Texas

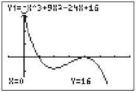
With the graph on the screen

2ND CALC TRACE to select the 'calc' menu.

1 to find where the x-value is zero.

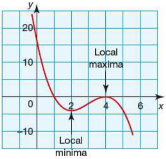
Type 0 and press ENTER.

The calculator gives the coordinate of the y-intercept.



Y1=-X^3+9X²-24X+16

X=0 Y=16



iv) Find the coordinates of the points where the graph has a local maxima or minima. Local maxima and minima refer to the peaks and dips of the graph respectively, i.e.

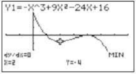
Casio

With the graph on the screen

G-Solv F5 to select the 'graph solve' menu.

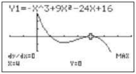
V-Window F3 to find any minimum points. The results are displayed in the graph.

G-Solv F5 Zoom F2 to find any maximum points.



Y1=-X^3+9X²-24X+16

dy/dx=0 MIN
X=2 Y=-4



Y1=-X^3+9X²-24X+16

dy/dx=0 MAX
X=4 Y=0

Texas

With the graph on the screen

2ND CALC F4 TRACE to select the 'calc' menu

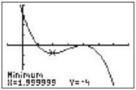
3 to find the local minima:

◀ ▶ to move the cursor to the left of the minimum point, ENTRY SOLVE ENTER to select the left bound.

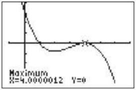
◀ ▶ to move the cursor to the right of the minimum point, ENTRY SOLVE ENTER to select the right bound.

ENTRY SOLVE ENTER to find the coordinates of the minimum point.

2ND CALC F4 TRACE 4 to search for the maximum point followed by the same procedure described above for the minimum point.



Minimum
X=1.999999 Y=-4



Maximum
X=4.0000012 Y=0

Note: Sometimes the solutions are only an approximation and not exact.

v) Complete the following results table for the coordinates of some points on the graph.

x	−1	0	1	2	3	4	5
f(x)							

The graphics calculator can produce a table of results for a given function, within a given range of values of x. This is shown below:

Casio

With the graph on the screen

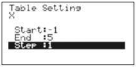
to select the table menu

to select the table settings and enter the properties as shown opposite.

to display the table of results.

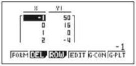
Once the table is displayed the remaining results can be viewed using


.



```
Table Setting
X

 Start:-1
 End  :5
 Step :1
```



```
    X     Y1
   -1     50
    0     16
    1      0
    2     -4
               -1
FORM DEL ROW EDIT G-CON G-PLT
```

Texas

With the graph on the screen

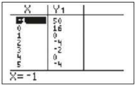
to select the table setup and enter the table properties as shown opposite.

to display the table of results

Once the table is displayed the remaining results can be viewed using

.

```
TABLE SETUP
 TblStart=-1
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```



```
  X    | Y1
  -1   | 50
  0    | 16
  1    | 0
  2    | -4
  3    | -2
  4    | 0
  5    | -4
X=-1
```

The instructions shown in the examples above will be useful when checking your solutions to various exercises throughout this textbook.

Exercise 3.9

Use a graphics calculator to help you to answer the following questions.

1. a) Sketch a graph of the function $y = x^3 - 7x + 6$
b) Find the roots of the function and label them clearly on your sketch.
c) Find where the graph intersects the y-axis. Label this clearly on your sketch.
d) Find the coordinates of any local maxima or minima.

2. a) Sketch a graph of the function $y = 2x^3 - 8x^2 - 22x - 12$
b) Find the roots of the function and label them clearly on your sketch.
c) Find where the graph intersects the y-axis. Label this clearly on your sketch.
d) Find the coordinates of any local maxima or minima.
e) Copy and complete the results table below, for the coordinates of some points on the graph.

x	–3	–2	–1	0	1	2	3	4	5	6	7
y											

3. a) Sketch a graph of the function $y = (x + 4)^2(x - 2)^2$
b) Find the roots of the function and label them clearly on your sketch.
c) Find where the graph intersects the y-axis. Label this clearly on your sketch.
d) Find the coordinates of any local maxima or minima.
e) Copy and complete the results table below, for the coordinates of some points on the graph.

x	–5	–4	–3	–2	–1	0	1	2	3
y									

4. Repeat Q.3 above for the function $y = (x + 4)^2(x - 2)^2 - 14$

5. a) Sketch a graph of the function $y = -(\frac{1}{2}x - 1)^2(x + 1)^2$
b) Find the roots of the function and label them clearly on your sketch.
c) Find where the graph intersects the y-axis. Label this clearly on your sketch.
d) Find the coordinates of any local maxima or minima.
e) Copy and complete the results table below, for the coordinates of some points on the graph.

x	–4	–3	–2	–1	0	1	2	3
y								

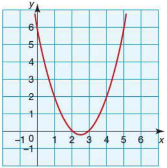
Use your graphics calculator to solve the following equations:

6. $2^x - 1 = \frac{1}{x^3}$ **7.** $3^{\frac{1}{2}x} = \frac{1}{4}x^4$ **8.** $\frac{1}{3^x} = -x^2 + \quad x + \frac{1}{2}$

SECTION 6

Finding a quadratic function from key values

Using factorised form



It is not necessary to see the graph of a quadratic function or to know the coordinates of a large number of points on the curve in order to determine its equation. All that is needed are the coordinates of certain key points.

In Topic 2 you learnt how to factorise a quadratic expression and therefore also a quadratic function. Factorised forms of a quadratic function give key information about its properties.

e.g. $f(x) = x^2 - 5x + 6$ can be factorised to give $f(x) = (x - 3)(x - 2)$.

The graph of the function is shown on the left.

There are features of the graph that relate directly to the equation of the function. Exercise 3.10 looks at these relationships.

Exercise 3.10

1. For each of the following quadratics:

i) write the function in factorised form

ii) with the aid of a graphics calculator if necessary, sketch the function and identify clearly where it crosses both axes.

a) $f(x) = x^2 + 3x + 2$

b) $f(x) = x^2 + 3x - 4$

c) $f(x) = x^2 + 3x - 10$

d) $f(x) = x^2 + 13x + 42$

e) $f(x) = x^2 - 9$

f) $f(x) = x^2 - 64$

2. Using your solutions to Q.1, describe any relationship that you can see between a function written in factorised form and its graph.

Exercise 3.10 shows that there is a direct link between a function written in factorised form and where its graph crosses the x-axis. The reason for this is explained using the earlier example of $y = (x - 3)(x - 2)$.

You know that where a graph crosses the x-axis, the y-coordinates are zero.

Hence $(x - 3)(x - 2) = 0$.

To solve the equation, either $(x - 2) = 0$ or $(x - 3) = 0$, and therefore $x = 2$ or $x = 3$ respectively. These are the x-coordinates of the points where the graph crosses the x-axis.

Worked examples

a) The graph of a quadratic function of the form $f(x) = x^2 + bx + c$ crosses the x-axis at $x = 4$ and $x = 5$. Determine the equation of the quadratic and state the values of b and c.

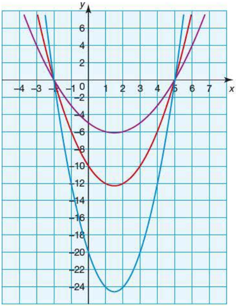
As the graph intercepts the x-axis at 4 and 5, the equation of the quadratic can be written in factorised form as $f(x) = (x - 4)(x - 5)$.

When expanded, the equation is written as $f(x) = x^2 - 9x + 20$.

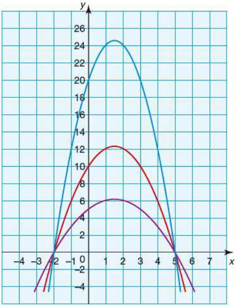
Therefore $b = -9$
$c = 20$

b) The graph of a quadratic function of the form $f(x) = ax^2 + bx + c$ crosses the x-axis at $x = -2$ and $x = 5$. It also passes through the point $(0, -20)$. Determine the equation of the quadratic and state the values of a, b and c.

This example is slightly more difficult than the first one. Although the graph crosses the x-axis at -2 and 5, this does not necessarily imply that the quadratic function is $f(x) = (x + 2)(x - 5)$. This is because more than one quadratic can pass through the points $x = -2$ and $x = 5$ as shown:



or



However, it is also known that the graph passes through the point $(0, -20)$. By substituting this into the equation $y = (x + 2)(x - 5)$ it can be seen that the equation is incorrect:

$$-20 \neq (0 + 2)(0 - 5)$$
$$-20 \neq -10$$

This implies that the coefficient of x^2, a, is not equal to 1 in the function $f(x) = ax^2 + bx + c$.

However because all possible quadratics that intersect the x-axis at $x = -2$ and $x = 5$ are stretches of $y = (x + 2)(x - 5)$ parallel to the y-axis, the quadratic must take the form $y = a(x + 2)(x - 5)$.

The value of a can be calculated by substituting the coordinate $(0, -20)$ into this equation.

$$\begin{aligned}\text{Therefore } -20 &= a(0+2)(0-5)\\ -20 &= -10a\\ a &= 2\end{aligned}$$

The quadratic function can therefore be written as $f(x) = 2(x+2)(x-5)$ which, when expanded, becomes $f(x) = 2x^2 - 6x - 20$.

Therefore $a = 2, b = -6, c = -20$

Exercise 3.11

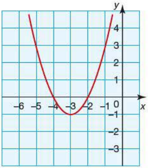
1. Find the equation of the quadratic graphs that intersect the x-axis at each of the following points. Give your answers in the form $f(x) = x^2 + bx + c$ clearly stating the values of b and c.

   a) $x = 0$ and $x = 2$ b) $x = -1$ and $x = -6$
   c) $x = -3$ and 4 d) $x = -\frac{1}{2}$ and 3

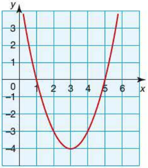
2. The graphs of six quadratic functions are shown below. In each case the function takes the form $f(x) = ax^2 + bx + c$ where $a = \pm 1$.

   From the graphs, find the equation and state clearly the values of a, b and c.

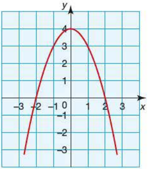
a)



b)



c)



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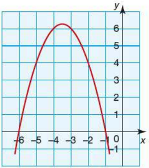

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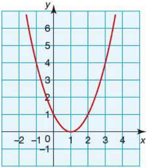

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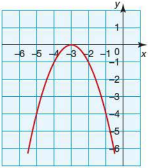
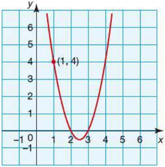
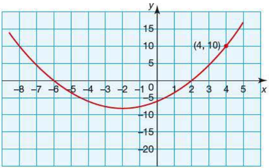


3. Find the equations of the quadratic functions graphed below, giving your answers in the form $f(x) = ax^2 + bx + c$. In each case the graph shows where the function intersects the x-axis and the coordinates of one other point on the graph.

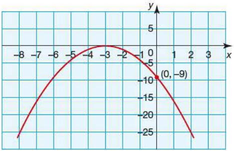
a)



b)



c)



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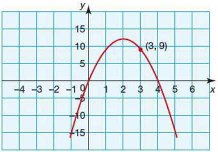

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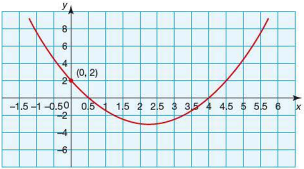

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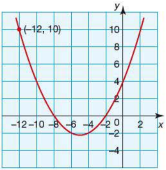
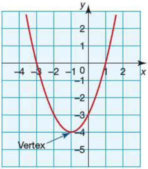
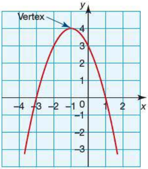


Using the vertex to find the quadratic function

The vertex of a quadratic function refers to the point at which the graph of the function is either a maximum or a minimum as shown below:



or



The coordinates of the vertex are useful for finding the quadratic function. You saw in Topic 2 how to transform functions. You may also have studied the extension material on how to factorise a quadratic by the method of completing the square. Both techniques are useful when deriving the quadratic function from the coordinates of its vertex.

Worked example A quadratic function is given as $f(x) = (x + 1)^2 - 9$.

i) Sketch the graph of the function by finding where it intersects each of the axes.

The intercept with the y-axis occurs when $x = 0$

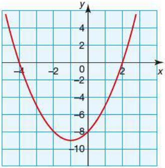
$$\begin{aligned} \text{Therefore } \; & f(0) = (0 + 1)^2 - 9 \\ \Rightarrow\; & f(0) = 1^2 - 9 \\ \Rightarrow\; & f(0) = -8 \end{aligned}$$

i.e. the y-intercept occurs at the coordinate $(0, -8)$

Intercepts with the x-axes occur when $y = 0$

$$\begin{aligned} \text{Therefore } \; (x + 1)^2 - 9 &= 0 \\ \Rightarrow \quad (x + 1)^2 &= 9 \\ \Rightarrow \quad x + 1 &= \pm\sqrt{9} \\ \Rightarrow \quad x + 1 &= \pm 3 \\ \Rightarrow \quad x &= 2 \text{ or } -4 \end{aligned}$$

The function can therefore be sketched:



ii) Find the coordinates of the graph's vertex.

There are two methods of approaching this.

Method 1

Due to symmetry, the x-coordinate of the vertex must be midway between the points where the graph intersects the x-axis.

Therefore the x-coordinate of the vertex is –1.

To find the y-coordinate of the vertex, substitute $x = -1$ into the function $f(x) = (x + 1)^2 - 9$:

$$f(-1) = (-1 + 1)^2 - 9$$
$$\Rightarrow f(-1) = -9$$

Therefore the coordinates of the vertex are (–1, –9).

This can be checked using the graphics calculator to find the coordinates of the minimum point.

Extension

Method 2

Look at the function written in completed square form as a series of transformations of $f(x) = x^2$.

The transformation that maps $f(x) = x^2$ to $f(x) = (x + 1)^2 - 9$ is the translation $\begin{pmatrix} -1 \\ -9 \end{pmatrix}$. As $f(x) = x^2$ has a vertex at (0, 0), $f(x) = (x + 1)^2 - 9$ has a vertex at (–1, –9).

Exercise 3.12

1. In each of the following, the quadratic functions are of the form $f(x) = (x - h)^2 + k$.

i) Find where the graph of the function intersects each axis.
ii) Sketch the function.
iii) Find the coordinates of the vertex.

iv) Check your answers to Q.1(i–iii) using a graphics calculator.

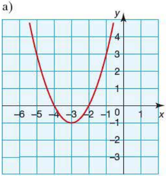
a) $f(x) = (x - 2)^2 - 9$
b) $f(x) = (x + 5)^2 - 1$
c) $f(x) = (x - 3)^2 - 4$
d) $f(x) = (x - \frac{1}{2})^2 - 16$
e) $f(x) = (x + 4)^2 - 10$
f) $f(x) = (x - 5)^2$

2. In each of the following, the quadratic function is of the form $f(x) = ax^2 + bx + c$, where $a = \pm 1$ and b and c are rational.

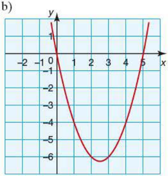
From the graph of each function:

i) find the equation of the quadratic in the form $f(x) = ax^2 + bx + c$
ii) find the coordinates of the vertex
iii) write the equation in the form $f(x) = a(x - h)^2 + k$
iv) expand your answer to Q.2(iii) and check it is the same as your answer to Q.2(i).

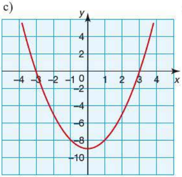
a)



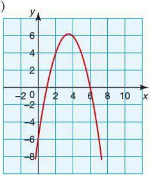
b)



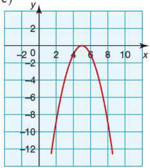
c)



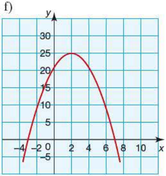
d)



e)



f)



3. a) Copy the table below and enter the relevant answers to Q.1 and Q.2 in the rows.

	In the form **$f(x) = a(x - h)^2 + k$**	**Coordinates of Vertex**
Example	$f(x) = (x + 1)^2 - 9$	$(-1, -9)$
1a		
1b		
1c		
1d		
1e		
1f		
2a		
2b		
2c		
2d		
2e		
2f		

b) Describe in your own words the relationship between the equation of a quadratic written in the form $f(x) = a(x - h)^2 + k$ and the coordinates of its vertex.

4. The following quadratics are of the form $f(x) = a(x - h)^2 + k$ where $a = \pm 1$ and h and k are rational. Find the coordinates of the vertex of each function.

a) $f(x) = (x - 2)^2 + 4$
b) $f(x) = (x + 5)^2 - 3$
c) $f(x) = (x - 6)^2 + 4$
d) $f(x) = x^2 - 3$
e) $f(x) = -(x + 5)^2 + 3$
f) $f(x) = -(x - 4)^2$

5. Sketch each of the quadratics in Q.4. Clearly label the coordinates of the vertex and where the graph intersects the y-axis.

6. The vertices of quadratic functions are given below. In each case the function is of the form $f(x) = ax^2 + bx + c$ where $a = 1$.

i) Work out, from the vertex, the equation in the form $f(x) = a(x - h)^2 + k$.

ii) Expand your equation from Q.6(i) to write the equation in the form $f(x) = ax^2 + bx + c$.

iii) Check your answers to parts (i) and (ii) using a graphics calculator.

a) $(-3, 6)$ b) $(2, -4)$

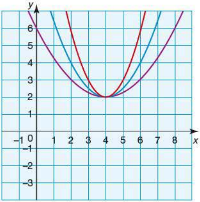
c) $(-1, 6)$ d) $(-4, 0)$

7. Repeat Q.6 where each of the quadratics are of the form $f(x) = ax^2 + bx + c$ and $a = -1$.

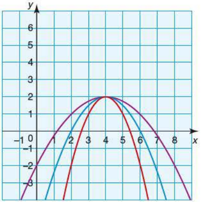
You will have seen that, in general, if a quadratic is written in the form $f(x) = a(x - h)^2 + k$, the coordinates of its vertex are (h, k).

Finding the equation of a quadratic function given a vertex and another point

It was shown earlier that if a quadratic is of the form $f(x) = ax^2 + bx + c$ and a can be any real number, then additional information is needed in order to deduce its exact equation. This is also the case regarding the coordinates of its vertex. If a can be any real number, then simply knowing the coordinates of the vertex is insufficient to deduce its equation, as more than one quadratic can be drawn with the same vertex as shown:



or



The coordinates of an additional point are also needed.

Worked example The graph of a quadratic function passes through the point $(-1, -25)$. Its vertex has coordinates $(2, 2)$.

i) Work out the equation of the quadratic in the form $f(x) = a(x-h)^2 + k$.

If the quadratic was of the form $f(x) = a(x-h)^2 + k$, the coordinates of its vertex suggest that its equation would be $f(x) = (x-2)^2 + 2$. However, the point $(-1, -25)$ does not fit this equation. Therefore the quadratic function is of the form $f(x) = a(x-h)^2 + k$ where $a \neq 1$.

The function must be written as $f(x) = a(x-2)^2 + 2$ and a can be calculated by substituting the values of the point $(-1, -25)$ into the function for x and y.

$$\begin{aligned} \text{Therefore} \quad -25 &= a(-1-2)^2 + 2 \\ \Rightarrow -25 &= 9a + 2 \\ \Rightarrow -27 &= 9a \\ \Rightarrow a &= -3 \end{aligned}$$

Therefore the quadratic function is $f(x) = -3(x-2)^2 + 2$.

ii) Find the equation of the quadratic in the form $f(x) = ax^2 + bx + c$ stating clearly the values of a, b and c.

The function $f(x) = -3(x-2)^2 + 2$ can be expanded to give:

$$\begin{aligned} f(x) &= -3(x^2 - 4x + 4) + 2 \\ \Rightarrow f(x) &= -3x^2 + 12x - 12 + 2 \\ \Rightarrow f(x) &= -3x^2 + 12x - 10 \end{aligned}$$

Therefore $a = -3$, $b = 12$ and $c = -10$.

iii) Work out where the graph intercepts the y–axis.

The graph intersects the y-axis when $x = 0$. Substituting $x = 0$ into the function $f(x) = -3x^2 + 12x - 10$ gives:

$$\begin{aligned} f(0) &= -3(0)^2 + 12(0) - 10 \\ f(0) &= -10 \end{aligned}$$

Therefore the graph intersects the y-axis at the point $(0, -10)$.

iv) Find the points of intersection with the x-axis, giving your answer in surd form.

The intercept with the x-axis occurs when $y = 0$. Substituting $y = 0$ into the equation gives:

$$0 = -3x^2 + 12x - 10$$

Using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a = -3$, $b = 12$ and $c = -10$

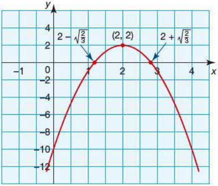
$$\text{Therefore } x = \frac{-12 \pm \sqrt{12^2 - 4(-3)(-10)}}{2(-3)}$$

$$x = \frac{-12 \pm \sqrt{144 - 120}}{-6} = \frac{-12 \pm \sqrt{24}}{-6}$$

$$x = \frac{-12}{-6} \pm \frac{\sqrt{24}}{-6} = 2 \pm \sqrt{\frac{24}{36}}$$

$$\text{Therefore } x = 2 \pm \sqrt{\frac{2}{3}}$$

v) Sketch the graph of the function.



Exercise 3.13

1. Find the coordinates of the vertex of each of the following quadratic functions:

a) $f(x) = 2(x - 1)^2 + 6$ b) $f(x) = 3(x + 4)^2 - 6$

c) $f(x) = 3x^2 - 4$ d) $f(x) = -2(x + 2)^2 - 6$

e) $f(x) = 2[(x - 6)^2 + 1]$ f) $f(x) = -\frac{1}{2}(x + 1)^2$

2. Four quadratic functions are given below. Each is written in the form $f(x) = a(x - h)^2 + k$ and its expanded form. Find the matching pairs.

a) $f(x) = (x - 3)^2 - 2$ b) $f(x) = 2(x + 1)^2 - 3$

c) $f(x) = 2x^2 - 4x + 3$ d) $f(x) = x^2 - 6x + 7$

e) $f(x) = 2x^2 + 4x$ f) $f(x) = 2(x - 1)^2 + 1$

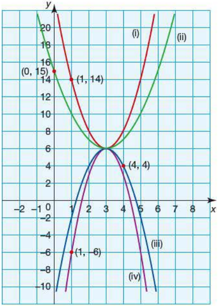
g) $f(x) = 2x^2 + 4x - 1$ h) $f(x) = 2[(x + 1)^2 - 1]$

3. The graphs of three of the four quadratic functions below pass through the point (1, 1). Which is the odd one out?

a) $f(x) = (x - 2)^2$ b) $f(x) = \frac{1}{2}(x + 3)^2 - 7$

c) $f(x) = 3(x - 3)^2 - 11$ d) $f(x) = (x - 1)^2 - 1$

4. The graph below shows four quadratic functions each with a vertex at (3, 6). The coordinates of one other point on each graph is also given.



Match each of the equations below with the correct graph.

a) $f(x) = (x-3)^2 + 6$ b) $f(x) = -2(x-3)^2 + 6$

c) $f(x) = 2\left[(x-3)^2 + 3\right]$ d) $f(x) = -3(x-3)^2 + 6$

5. The coordinates of the vertex of a quadratic function and one other point are given in parts (a)–(f) below. Work out the quadratic function:

i) in the form $f(x) = a(x-h)^2 + k$

ii) in the form $f(x) = ax^2 + bx + c$, stating clearly the values of a, b and c.

	Vertex	Other point
a)	(−1 ,−5)	(1, 3)
b)	(1, 4)	(2, 2)
c)	(−4, −4)	(−3, −1)
d)	(−5, −2)	(−3, 0)
e)	(3, 0)	(0, −9)
f)	(−5, −12)	(−7, −6)

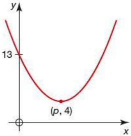
6. For each of the quadratic functions in Q.5:

i) find the y-intercept

ii) work out where/if the graph intersects the x-axis, giving your answer in surd form

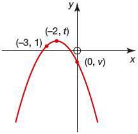
iii) check your answers using your graphics calculator.

7. The quadratic function shown below has a vertex at $(p, 4)$ where $p > 0$ and an intercept with the y-axis at $(0, 13)$.



If the function is of the form $f(x) = x^2 + bx + c$, find the value of p.

8. The quadratic function shown below is of the form $f(x) = ax^2 + bx + c$ where $a = -1$. The graph of the function has a vertex at $(-2, t)$ and it also passes through the point $(-3, 1)$.



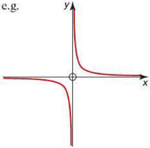
a) Find the value of t.
b) Work out the value of v, the y-intercept.

SECTION 7 Finding the equation of other common functions

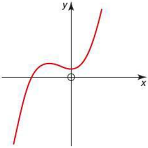
Section 6 showed that it is possible to find the equation of a quadratic function from some of its properties, such as the intercepts with the axes and the coordinates of the vertex.

Section 3 showed that it is possible to recognise the type of function by the shape of its graph.

e.g.



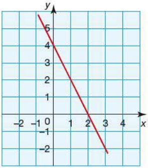
is a reciprocal function of the form $f(x) = \frac{a}{x}$ where $a \neq 0$



is a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

It is therefore possible to find the equation of a graph from its shape and the coordinates of some of the points that lie on the graph.

Worked examples



a) The following function passes through the points (0, 4) and (2, 0). Work out its equation.

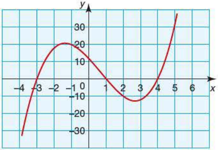
From the graph (left) we can recognise the function is linear, therefore it is of the form $f(x) = ax + b$, where a represents the gradient and b represents the y-intercept.

$$\text{Gradient} = \frac{4-0}{0-2} = \frac{4}{-2} = -2$$

y-intercept = 4

Therefore $f(x) = -2x + 4$.

b) The graph below has the equation $f(x) = x^3 + bx^2 - 11x + d$. It intercepts the y-axis at (0, 12) and the x-axis in three places, one of which is the point (4, 0).



Find the values of b and d in the function $f(x) = x^3 + bx^2 - 11x + d$.

At the intercept with the y-axis, $x = 0$. This can be substituted into the equation to work out d:

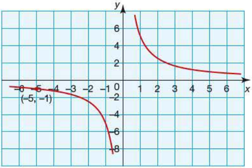
$$\begin{aligned} & f(0) = 0^3 + b(0)^2 - 11(0) + d \\ \Rightarrow\ & 12 = 0 + 0 - 0 + d \\ \Rightarrow\ & 12 = d \end{aligned}$$

When $x = 4$, $y = 0$. This can be substituted into the equation to find the value of b:

$$\begin{aligned} & f(4) = 4^3 + b(4)^2 - 11(4) + 12 \\ \Rightarrow\ & 0 = 64 + 16b - 44 + 12 \\ \Rightarrow\ & 0 = 32 + 16b \\ \Rightarrow\ & -32 = 16b \\ \Rightarrow\ & b = -2 \end{aligned}$$

Therefore $f(x) = x^3 - 2x^2 - 11x + 12$.

c) The function below passes through the point (–5, –1).



Given that the function is of the form $f(x) = \frac{a}{x}$ work out the value of a.

As its equation is of the form $f(x) = \frac{a}{x}$, the value of a can be calculated by substituting in the value of x and y of a point on the graph:

$$\begin{aligned} \Rightarrow\ & -1 = \frac{a}{-5}, \\ \Rightarrow\ & a = 5 \end{aligned}$$

Therefore $f(x) = \frac{5}{x}$.

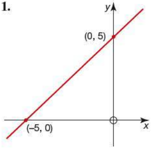
Exercise 3.14

In Q.1–8, linear functions take the form $f(x) = ax + b$, cubic functions take the form $f(x) = ax^3 + bx^2 + cx + d$ and reciprocal functions take the form $f(x) = \frac{a}{x}$.

For each question:

a) identify the type of function from the shape of the graph
b) find the values of the unknown coefficients a, b, c or d.

1.



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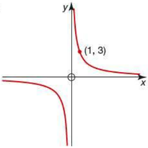
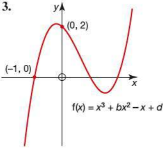


3.



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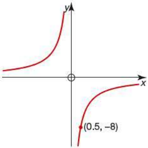

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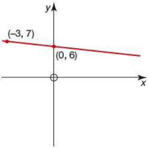

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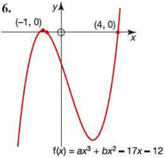

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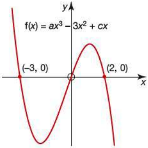

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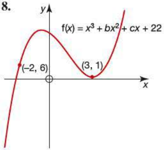
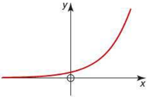
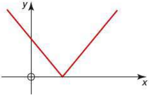


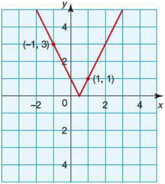
A similar method can be used to work out the equations of exponential and absolute functions. Section 3 showed that exponential functions take the form $f(x) = a^x$ where $0 < a < 1$ or $a > 1$ and in general have a graph as shown:



Linear absolute functions take the form $f(x) = |ax + b|$ and in general are as shown:

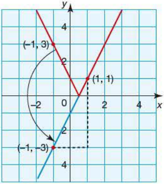


Worked examples **a)** The graph below is of the form $f(x) = |ax + b|$. It passes through the points $(-1, 3)$ and $(1, 1)$.



Find the values of a and b.

There are several ways of approaching this type of question. One is to work on the basis that the graph is a linear function rather than an absolute function.



The linear function takes the form $f(x) = ax + b$ and passes through $(1, 1)$ and $(–1, –3)$. The values of a and b can be calculated:

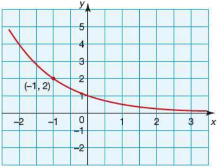
Gradient, $a = \frac{1 - -3}{1 - -1} = \frac{4}{2} = 2$

Substitute the point $(1, 1)$ into the equation to find the y-intercept:

$$y = ax + b$$
$$\Rightarrow 1 = 2 \times 1 + b$$
$$\Rightarrow b = -1$$

The linear function is therefore $f(x) = 2x - 1$, and the absolute function is $f(x) = |2x - 1|$.

b) The graph below passes through the point $(–1, 2)$. Identify the type of function and work out its equation.



From the shape of the graph, we can identify that the function is of the form $f(x) = a^x$ where $0 < a < 1$. To find the value of a, substitute the values of x and y of a point on the curve.

Substituting $(-1, 2)$ into $y = a^x$

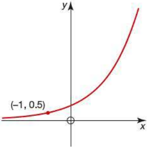
$$\begin{aligned} &\Rightarrow 2 = a^{-1} \\ &\Rightarrow 2 = \frac{1}{a} \\ &\Rightarrow a = \frac{1}{2} \end{aligned}$$

Therefore $f(x) = \left(\frac{1}{2}\right)^x$.

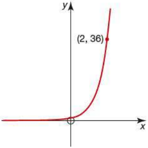
Exercise 3.15

1. Find the equation of the following functions:

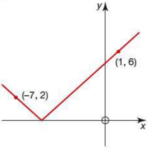
a)



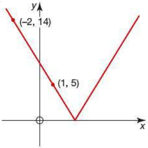
b)



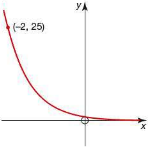
c)



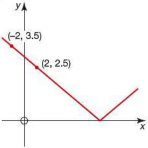
d)



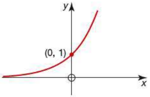
e)



f)

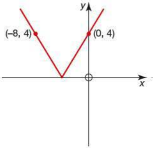


2. A graph of the form $f(x) = a^x$ is shown below:

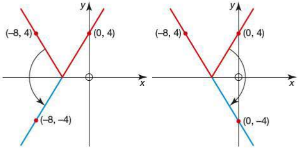


Explain clearly why the information given is insufficient to find the equation of the graph.

3. A graph of the form $f(x) = |ax + b|$ is shown below:



Explain clearly why it does not matter whether the absolute function is considered as either of the two linear functions below in order to find its equation.



SECTION 8

Composite functions

So far we have dealt with functions individually. However if two functions are a function of x, it is possible to combine them to create a further function of x. These are known as **composite** functions.

Worked examples

a) $f(x) = 3x$ and $g(x) = 2x - 1$

i) Write an expression for $f(g(x))$.

$f(g(x))$ means that $g(x)$ is substituted for x in the function $f(x)$,

i.e.

$$\begin{aligned} f(g(x)) &= f(2x - 1) \\ &= 3(2x - 1) \\ &= 6x - 3 \end{aligned}$$

ii) Write an expression for $g(f(x))$.

$g(f(x))$ means that $f(x)$ is substituted for x in the function $g(x)$,

i.e.

$$\begin{aligned} g(f(x)) &= g(3x) \\ &= 2(3x) - 1 \\ &= 6x - 1 \end{aligned}$$

It can be seen from the answers to parts(i) and (ii) that $f(g(x))$ and $g(f(x))$ are not equal.

b) $f(x) = x - 4$ and $g(x) = 2x + 1$

i) Write an expression equivalent to $f(g(x))$.

Substitute $g(x)$ for x in the function $f(x)$:

$$\begin{aligned} f(g(x)) &= f(2x + 1) \\ &= (2x + 1) - 4 \\ &= 2x - 3 \end{aligned}$$

ii) Evaluate $f(g(3))$.

From part (i), $f(g(x)) = 2x - 3$. Substitute $x = 3$ into the function $f(g(x))$.

Therefore

$$\begin{aligned} f(g(3)) &= 2 \times 3 - 3 \\ &= 3 \end{aligned}$$

iii) Solve $f(g(x)) = 0$,

i.e. find the value of x which produces $f(g(x)) = 0$.

Therefore solve

$$\begin{aligned} 2x - 3 &= 0 \\ \Rightarrow x &= \tfrac{3}{2} \end{aligned}$$

Exercise 3.16

1. For the function $f(x) = 3x + 4$, evaluate:

a) $f(0)$ b) $f(2)$ c) $f(-1)$

2. If $g(x) = \frac{1}{2}x + 2$, evaluate:

a) $f(4)$ b) $f(-4)$ c) $f\left(\frac{1}{2}\right)$

3. $f(x)$ and $g(x)$ are given for each part below. In each case, find a simplified expression for $f(g(x))$.

a) $f(x) = 2x$
$g(x) = 3x$

b) $f(x) = 4x$
$g(x) = x - 3$

c) $f(x) = 3x - 1$
$g(x) = 3x$

d) $f(x) = x - 2$
$g(x) = 3x + 1$

e) $f(x) = 2x - 2$
$g(x) = 3x + 1$

f) $f(x) = 2 - 2x$
$g(x) = x + 4$

4. $f(x) = 5x - 3$ and $g(x) = 2x + 3$
  a) Solve $f(x) = 0$.
  b) Solve $g(x) = 0$.
  c) Find the value of x where $f(x) = g(x)$.

5. $f(x) = 4x - 2$ and $g(x) = 2x - 4$
  a) Find a single expression for $f(g(x))$.
  b) Find a single expression for $g(f(x))$.
  c) i) Evaluate $f(g(2))$.
     ii) Evaluate $g(f(1))$.

6. $f(x) = 2x$ and $g(x) = 2x - 3$
  a) Write a single expression for:
     i) $f(g(x))$ ii) $g(f(x))$
  b) Find the value of:
     i) $f(g(0))$ ii) $g(f(0))$
  c) Explain whether, in this case, $f(g(x))$ can ever be equal to $g(f(x))$.

Worked example $f(x) = x^2 - 2$ and $g(x) = x + 1$

i) Find a simplified expression for $f(g(x))$.

$g(x)$ is substituted for x in the function $f(x)$:

$$\begin{aligned} f(x+1) &= (x+1)^2 - 2 \\ &= x^2 + 2x + 1 - 2 \\ &= x^2 + 2x - 1 \end{aligned}$$

ii) Evaluate $f(g(2))$.

$f(g(x)) = x^2 + 2x - 1$

$$\begin{aligned} \text{So } f(g(2)) &= (2)^2 + 2(2) - 1 \\ &= 4 + 4 - 1 \\ &= 7 \end{aligned}$$

iii) Solve $f(g(x)) = 0$ giving your answer in exact form.

$$x^2 + 2x - 1 = 0$$

As the quadratic does not factorise it can be solved either by completing the square of using the quadratic formula. Using the quadratic formula gives:

$$x = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$x = -1 \pm \sqrt{2}$$

Exercise 3.17

1. $f(x)$ and $g(x)$ are given for each part below. In each case find a simplified expression for $f(g(x))$.

a) $f(x) = x^2$ $\quad g(x) = x + 1$
b) $f(x) = 2x^2$ $\quad g(x) = x - 1$
c) $f(x) = x^2 + 1$ $\quad g(x) = 2x$
d) $f(x) = x^2 - 1$ $\quad g(x) = 2x + 1$
e) $f(x) = 2x^2$ $\quad g(x) = 3x - 2$
f) $f(x) = 4x^2$ $\quad g(x) = \frac{1}{2}x + 1$
g) $f(x) = x^2$ $\quad g(x) = 3 - x$
h) $f(x) = \frac{1}{2}x^2$ $\quad g(x) = 2 - 3x$

2. $f(x)$ and $g(x)$ are given for each part below.

i) Find the simplified form for $f(g(x))$.
ii) Evaluate $f(g(0))$.
iii) Evaluate $f(g(1))$.
iv) Evaluate, where possible, $f(g(-1))$.

a) $f(x) = \dfrac{3}{x}$ $\quad g(x) = x + 1$
b) $f(x) = x^3$ $\quad g(x) = 2x - 1$
c) $f(x) = 2^x$ $\quad g(x) = 3x$
d) $f(x) = |x + 1|$ $\quad g(x) = 4 - x$

3. $f(x) = x^2 - 3$ and $g(x) = x + 1$

a) Find $f(g(x))$.
b) Evaluate $f(g(-1))$.
c) Solve $f(g(x)) = 0$ leaving your answer in surd form.

4. $f(x) = 2x^2$ and $g(x) = x - 1$

a) Find $f(g(x))$.
b) Evaluate $f(g(2))$.
c) Solve $f(g(x)) = 18$.

SECTION 9

Inverse functions

The **inverse** of a function is its reverse, i.e. it 'undoes' the function's effects. The inverse of the function $f(x)$ is written as $f^{-1}(x)$.

Worked examples

a) Find the inverse of the following functions:

i) $f(x) = x + 2$

This can be done by following these steps.

Write the equation in terms of y: $\quad y = x + 2$
Swap x and y: $\quad x = y + 2$
Rearrange to make y the subject: $\quad y = x - 2$
Therefore $f^{-1}(x) = x - 2$.

ii) $g(x) = 2x - 3$

Write the equation in terms of y: $y = 2x - 3$

Swap x and y: $x = 2y - 3$

Rearrange to make y the subject: $x + 3 = 2y \Rightarrow y = \frac{x+3}{2}$

Therefore $g^{-1}(x) = \frac{x+3}{2}$.

b) If $f(x) = \frac{x-3}{3}$, calculate:

i) $f^{-1}(2)$

$f^{-1}(x) = 3x + 3$

$f^{-1}(2) = 9$

ii) $f^{-1}(-3)$

$f^{-1}(x) = 3x + 3$

$f^{-1}(-3) = -6$

Exercise 3.18

Find the inverse of each of the following functions:

1. a) $f(x) = x + 3$ b) $f(x) = x + 6$
c) $f(x) = x - 5$ d) $g(x) = x$
e) $h(x) = 2x$ f) $p(x) = \frac{x}{3}$

2. a) $f(x) = 4x$ b) $f(x) = 2x + 5$
c) $f(x) = 3x - 6$ d) $f(x) = \frac{x+4}{2}$
e) $g(x) = \frac{3x-2}{4}$ f) $g(x) = \frac{8x+7}{5}$

3. a) $f(x) = \frac{1}{2}x + 3$ b) $g(x) = \frac{1}{4}x - 2$
c) $h(x) = 4(3x - 6)$ d) $p(x) = 6(x + 3)$
e) $q(x) = -2(-3x + 2)$ f) $f(x) = \frac{2}{3}(4x - 5)$

Exercise 3.19

1. If $f(x) = x - 4$, evaluate:
a) $f^{-1}(2)$ b) $f^{-1}(0)$ c) $f^{-1}(-5)$

2. If $f(x) = 2x + 1$, evaluate:
a) $f^{-1}(5)$ b) $f^{-1}(0)$ c) $f^{-1}(-11)$

3. If $g(x) = 6(x - 1)$, evaluate:
a) $g^{-1}(12)$ b) $g^{-1}(3)$ c) $g^{-1}(6)$

4. If $g(x) = \frac{2x+4}{3}$, evaluate:
a) $g^{-1}(4)$ b) $g^{-1}(0)$ c) $g^{-1}(-6)$

5. If $h(x) = \frac{1}{3}x - 2$, evaluate:
a) $h^{-1}(-\frac{1}{2})$ b) $h^{-1}(0)$ c) $h^{-1}(-2)$

6. If $f(x) = \frac{4x-2}{5}$, evaluate:
a) $f^{-1}(6)$ b) $f^{-1}(-2)$ c) $f^{-1}(0)$

SECTION 10

Logarithmic functions

Log buttons

Earlier in this topic exponential functions were investigated, i.e. functions of the form $y = a^x$ where the variable x is the exponent (power). Solving exponential equations was done primarily using knowledge of indices.

e.g. Solve $2^x = 32$

$\Rightarrow x = 5$ because $2^5 = 32$

However had x not been an integer value, the solution would have been more difficult to find. This section looks at the inverse of the exponential function.

Your calculator has a logarithm button. This will be used throughout this section.

Casio	Texas
	

Exercise 3.20

1. Use the log button on your calculator to evaluate:
   a) log 1
   b) log 10
   c) log 100
   d) log 1000
2. Explain clearly, referring to your answers to Q.1, what you think the log button does.
3. a) The log of what number will give an answer of –1?
   b) Explain your answer to part (a).

You will have concluded from the exercise above that the log button is related to powers of 10,

i.e. $\log_{10} 100 = 2 \Leftrightarrow 10^2 = 100$

↑

The base

Logarithms can have a base other than 10, but the relationship still holds,

i.e. $\log_5 125 = 3 \Leftrightarrow 5^3 = 125$

In the example above, the relationship can be explained as the logarithm (3) of a positive number (125) is the power (3) to which the base (5) must be raised to give that number.

In general, therefore, $\log_a y = x \Leftrightarrow a^x = y$.

This generates three important results:

1. $\log_a a = 1$ e.g. $\log_7 7 = 1$ because $7^1 = 7$

2. $\log_a 1 = 0$ e.g. $\log_7 1 = 0$ because $7^0 = 1$

3. $\log_a \left(\frac{1}{a}\right) = -1$ e.g. $\log_7 \left(\frac{1}{7}\right) = -1$ because $7^{-1} = \left(\frac{1}{7}\right)$

Note: The log button on your calculator means log to the base 10, i.e. $\log_{10} x$.

Worked examples

a) Express $27 = 3^3$ in logarithmic notation.

$$\log_3 27 = 3$$

b) Express $\log_2 128 = 7$ in index notation.

$$2^7 = 128$$

c) Solve the equation $\log_4 1024 = x$.

$$4^x = 1024$$
$$\Rightarrow x = 5$$

d) Evaluate x in the equation $\log_{16} 2 = x$.

$$16^x = 2$$
$$\Rightarrow (2^4)^x = 2 \Rightarrow 2^{4x} = 2$$
$$\Rightarrow 4x = 1 \Rightarrow x = \tfrac{1}{4}$$

Exercise 3.21

1. Express the following using logarithmic notation:

a) $2^3 = 8$ b) $7^3 = 343$ c) $10^5 = 100\,000$

d) $3^0 = 1$ e) $2^{-2} = \frac{1}{4}$ f) $p^4 = q$

2. Express the following using index notation:

a) $\log_2 64 = 6$ b) $\log_3 81 = 4$ c) $\log_5 625 = 4$

d) $\log_3 \frac{1}{9} = -2$ e) $\log_x 1 = 0$ f) $\log_5 \frac{1}{125} = -3$

3. Solve the following equations:

a) $\log_{10} 10 = x$ b) $\log_{10} 10^6 = y$ c) $\log_5 3125 = a$

d) $\log_7 1 = b$ e) $\log_2 x = 5$ f) $\log_3 x = -3$

g) $\log_5 m = -1$ h) $\log_9 n = \frac{1}{2}$

Because a logarithm is an index (power), it is governed by the same laws as indices. The three basic laws of logarithms, which work for all bases, are as follows:

1. $\log_a x + \log_a y = \log_a (xy)$

e.g. $\log_{10} 5 + \log_{10} 3 = \log_{10} (5 \times 3) = \log_{10} 15$

2. $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

e.g. $\log_{10} 12 - \log_{10} 4 = \log_{10} \left(\frac{12}{4}\right) = \log_{10} 3$

3. $\log_a x^y = y\log_a x$

e.g. $\log_{10} 3^2 = 2\log_{10} 3$

Below is a mathematical proof to show why the first law is true:

To prove that $\log_a x + \log_a y = \log_a (xy)$, let $p = \log_a x$ and let $q = \log_a y$.

Therefore, $x = a^p$ and $y = a^q$

so $xy = a^p \times a^q = a^{p+q}$.

Substituting a^{p+q} for xy in $\log_a (xy)$ gives: $\log_a a^{p+q}$

This simplifies to: $p + q$

Therefore $\log_a (xy) = p + q$

But, $p = \log_a x$ and $q = \log_a y$.

Therefore $\log_a x + \log_a y = \log_a (xy)$

Similar proofs can be found for the other two laws of logarithms.

Worked examples

a) Express $3\log_{10} 2 - \log_{10} 16$ as a single logarithm.

Using the laws of logarithms, each term can be rewritten to give:

$$\log_{10} 2^3 - \log_{10} 16$$

$$= \log_{10} 8 - \log_{10} 16 \qquad = \log_{10}\left(\frac{8}{16}\right) \qquad = \log_{10}\left(\frac{1}{2}\right)$$

b) Express $3 + \log_{10} 5$ as a single logarithm.

To combine the two terms each needs to be written in log form with the same base.

3 can be written in terms of logs as $\log_{10} 1000$

$$\begin{aligned}\text{Therefore } 3 + \log_{10} 5 &= \log_{10} 1000 + \log_{10} 5\\ &= \log_{10} (1000 \times 5)\\ &= \log_{10} 5000\end{aligned}$$

c) Simplify $\dfrac{\log_{10} 32}{\log_{10} 128}$.

Both 32 and 128 are powers of 2. Written in terms of powers of 2, the fraction can be written as $\dfrac{\log_{10} 2^5}{\log_{10} 2^7}$

$$= \frac{5\log_{10} 2}{7\log_{10} 2} = \frac{5}{7}$$

Exercise 3.22

1. Express each of the following as a single logarithm:

a) $\log_{10} 2 + \log_{10} 5$

b) $\log_{10} 8 - \log_{10} 4$

c) $\log_{10} 3 - \log_{10} 1$

d) $\log_{10} 6 + \log_{10} 3 - \log_{10} 2$

e) $2\log_{10} 4 - 2\log_{10} 2 + 3\log_{10} 1$

f) $\frac{1}{4}\log_{10} a - \frac{1}{4}\log_{10} b$

g) $1 - \log_{10} 5$

h) $\log_{10} \sqrt{x} - 2\log_{10} x$

i) $3\log_{10} \sqrt[3]{a} - \frac{1}{2}\log_{10} a$

j) $3\log_{10} a - \frac{1}{2}\log_{10} b + 2$

2. Simplify each of the following:

a) $\log_{10} 100$

b) $2\log_4 64$

c) $\frac{1}{2}\log_{10} 4$

d) $\frac{1}{4}\log_{10} 625$

e) $-\frac{1}{3}\log_{10} 27$

f) $\dfrac{\log_{10} 64}{\log_{10} 16}$

g) $\dfrac{3\log_{10} 3}{\log_{10} 27}$

h) $\dfrac{\log_{10} \sqrt{a}}{2\log_{10} a}$

Solving exponential equations using logarithms

Logarithms can be used to solve more complex exponential equations than the ones dealt with so far.

e.g. To solve the exponential equation $3^x = 12$ is problematic because the solution is not an integer value. We know that the solution must lie between $x = 2$ and $x = 3$ because $3^2 = 9$ and $3^3 = 27$. By trial and error the solution could be found to one decimal place, however this is a laborious process and not practical, especially if the solution is required to two or more decimal places.

However, the equation can be solved using logs.

$$3^x = 12$$

Take logs of both sides of the equation: $\log_{10} 3^x = \log_{10} 12$

Using the law $\log_a x^y = y\log_a x$: $x\log_{10} 3 = \log_{10} 12$

Divide both sides by $\log_{10} 3$: $x = \dfrac{\log_{10} 12}{\log_{10} 3}$

The solution can either be left in the exact form $x = \dfrac{\log_{10} 12}{\log_{10} 3}$ or given to the required number of decimal places or significant figures, e.g. $x = 2.26$ (3 s.f.).

Worked examples

a) Solve the equation $5^x = 100$, giving your answer to 3 s.f.

$$\begin{aligned} 5^x &= 100 \\ \Rightarrow \log_{10} 5^x &= \log_{10} 100 \\ \Rightarrow x\log_{10} 5 &= 2 \\ \Rightarrow \quad x &= \frac{2}{\log_{10} 5} = 2.86 \end{aligned}$$

b) Solve the equation $\log_3 x = 4$.

$$\begin{aligned} \Rightarrow \log_3 x &= 4 \\ \Rightarrow \quad 3^4 &= x \\ \Rightarrow \quad x &= 81 \end{aligned}$$

c) Find the smallest integer value of x such that $5^x > 100000$. Firstly solve the equation $5^x = 100000$ to find the critical value of x:

$$\begin{aligned} 5^x &= 100000 \\ \Rightarrow \log_{10} 5^x &= \log_{10} 100000 \\ \Rightarrow x\log_{10} 5 &= 5 \\ \Rightarrow \quad x &= \frac{5}{\log_{10} 5} = 7.15 \text{ (3 s.f.)} \end{aligned}$$

Therefore the smallest integer value of x which satisfies the inequality $5^x > 100000$ is 8.

Exercise 3.23

1. Solve these equations. Give decimal answers correct to 3 s.f.

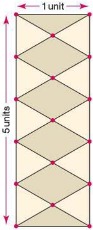
a) $4^x = 50$
b) $9^a = 34$
c) $3^y = \frac{1}{5}$
d) $\left(\frac{1}{10}\right)^p = 4$
e) $5^{x+1} = 100$
f) $8^{y-6} = 500$
g) $7^{2x+1} = 6000$
h) $5^{-m} = 20$
i) $\log_5 x = 6$
j) $\log_2 b = 5$
k) $3\log_{10} x = \log_{10} 2$
l) $\frac{1}{2}\log_{10} x = \log_{10} 6 - 1$

2. Solve each of the following inequalities:

a) $6^x > 1000$
b) $5^y > 10$
c) $2^{x+1} > 50$
d) $0.3^{2x+1} < 8$

3. Find the smallest integer x such that $3^{\frac{1}{2}x-2} > 120$.

4. Find the largest integer m such that $0.4^m > \frac{1}{1000}$.



Paving blocks

Isometric dot paper is needed for this investigation.

A company offers to block pave driveways for homeowners. The design is always rectangular. An example is shown left:

The design shown is 1 unit by 5 units.

It consists of three different shapes of block paving:

10 pieces in the form of an equilateral triangle
2 pieces in the form of an isosceles triangle
4 pieces in the form of a rhombus

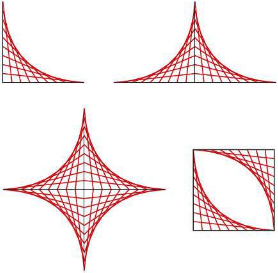
Different sized rectangular designs will have different numbers of each of the different shaped blocks.

1. Draw a design of width 2 units and height 4 units. Count the number of each type of block.
2. Draw at least six different sized rectangular designs, each time counting the number of different shaped blocks.
3. Enter your results in a table similar to the one shown:

Dimensions		Number of blocks		
Width	**Height**	**Equilateral**	**Isosceles**	**Rhombus**
1	5	10	2	4

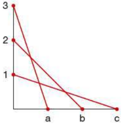
4. Investigate, by drawing more designs if necessary, the relationship between the width and height of a design and the number of different blocks needed.
5. a) Describe in words the relationship between the width (w) and the number of isosceles triangles (i).
   b) Write your rule from Q.5(a) using algebra.
6. a) Describe in words the relationship between the height (h) and the number of equilateral triangles (e).
   b) Write your rule from Q.6(a) using algebra.
7. a) Describe in words the relationship between the width, the height and the number of rhombuses (r).
   b) Write your rule from Q.7(a) using algebra.

Regions and intersections



The patterns above are examples of 'curve stitching'. Although the patterns produce a curved effect, all the lines used are in fact straight.

Below is how to construct a simple one:



Lines are drawn from points
$a \to 3$
$b \to 2$
$c \to 1$

This 3 × 3 curve stitch has produced three points of intersection and six enclosed regions.

Different sized curve stitches will produce a different number of points of intersection and a different number of enclosed regions.

1. Investigate the number regions and points of intersection for different sized curved stitching patterns. Record your results in a table similar to the one shown below:

Dimension of curve stitch	Number of points of intersection	Number of enclosed regions
1 × 1		
2 × 2		
3 × 3	3	6
4 × 4		
etc		

2. For an $m \times m$ pattern, write an algebraic rule for the number of points of intersection p.
3. For an $m \times m$ pattern, write an algebraic rule for the number of enclosed regions r.

Modelling: Parking error

A driver parks his car on a road at the top of a hill with a constant gradient. Unfortunately he does not put the handbrake on properly and as a result the car starts to roll down the hill.

The incident was captured on CCTV, so the distance travelled (m) and time (s) were both recorded. These results are presented in the table below:

Time (s)	0	2	4	6	8	10	12
Distance rolled (m)	0	0.2	0.8	1.8	3.2	5	7.2

1. Plot a graph of the data, with time on the x-axis and distance on the y-axis and draw a curve through the points.
2. Describe the relationship between the time and distance rolled.
3. Find the equation of the curve.
4. Use your equation to predict how far the car will have rolled after 15 seconds.
5. If the road is 120 m long, use your equation to estimate how long it will take the car to reach the bottom of the hill.

ICT Activity

A type of curve you have not encountered on this course is a rectangular hyperbola.

The equation of a rectangular hyperbola is given as $x^2 - y^2 = a^2$ where a is a constant.

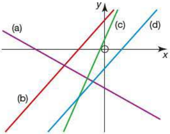
1. Investigate, using the internet if necessary, what a hyperbola is and how it relates to cones.
2. a) Rearrange the equation of a rectangular hyperbola to make y the subject.
   b) By letting $a = 1$, plot a graph of a rectangular hyperbola using a graphics calculator.
   c) Sketch the graph.
   d) By changing the value of a, determine its effect on the shape of the graph.
3. Using your graphs of a rectangular hyperbola as reference, determine the equation of any asymptotes.

SECTION 12 Student assessments

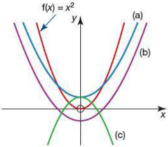
Student assessment 1

1. Calculate the range of these functions:
   a) $f(x) = 3x - x^3$; $-3 \leqslant x \leqslant 3$
   b) $f(x) = 2x^2 - 4$; $x \in \mathbb{R}$
2. Linear functions take the form $f(x) = ax + b$. Explain the effect a and b have on the shape and/or position of the graph.
3. The diagram below shows four linear functions. These are $y = x + 3$, $y = x - 2$, $y = -\frac{1}{2}x - 4$ and $y = 2x + 1$.

   State, giving reasons, which line corresponds with which function.

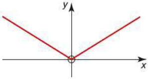


4. The diagram below shows four quadratic functions. The function $f(x) = x^2$ is highlighted.



Give possible equations for each of the quadratic curves (a), (b) and (c).

5. The graph of the absolute function $f(x) = |x|$ is shown below:

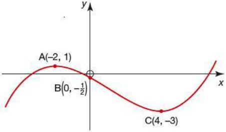


Copy the graph. Sketch and clearly label the following absolute functions on the same axes:

a) $f(x) = |2x| + 2$ b) $f(x) = |x - 2|$

Student assessment 2

1. The graph below shows the function $y = f(x)$:
Three points are labelled: A(–2, 1), B($0, -\frac{1}{2}$), C(4, –3).



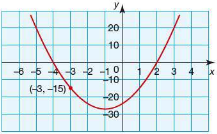
Give the coordinates of the point's images when mapped to the following:

a) $y = f(x) - 3$ b) $y = 3f(x)$

2. Sketch the graph of $y = x^2$. On the same axes, sketch, using transformations of graphs, both of the following. Label each graph clearly.
   a) $y = (x - 4)^2$ b) $y = \frac{1}{2}x^2$
3. Sketch the graph of $y = \frac{1}{x}$. On the same axes, sketch, using transformations of graphs, both of the following. Label each graph clearly.
   a) $y = \frac{3}{x}$ b) $y = \frac{1}{x - 4}$
   c) i) Give the equation of any asymptotes in Q.3(a).
      ii) Give the equation of any asymptotes in Q.3(b).
4. A function is given as $f(x) = x^3 + 6x^2 - 15x + 10$. Use your graphics calculator to find:
   a) the coordinates of the points where the function has a local maxima and minima
   b) the root(s) of the function
   c) the value of the y-intercept.

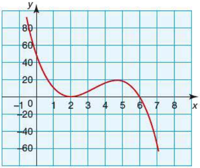
Student assessment 3

1. The graph of a quadratic function of the form $f(x) = x^2 + bx + c$ intercepts the x-axis at $x = -3$ and $x = 2$. Determine the equation of the quadratic and state the values of b and c.
2. Deduce the equation of the quadratic function shown below with roots at $x = -4$ and $x = 2$ which passes through the point $(-3, -15)$.
   Give your answer in the form $f(x) = ax^2 + bx + c$.



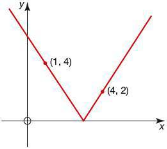
3. Find the coordinates of the vertices of these quadratic functions written in the form $y = a(x - h)^2 + k$.
   a) $y = (x + 1)^2 - 3$ b) $y = 3(x - 5)^2 + 1$

4. The cubic function below is of the form $f(x) = ax^3 + bx^2 + cx + d$. Its roots are at $x = 6$ and $x = 2$.



If the value of $a = -2$ and the graph intersects the y-axis at +48, determine the values of b, c and d.

5. Calculate the equation of the function graphed below which passes through the points (1, 4) and (4, 2).



Student assessment 4

1. $f(x) = 4x$ and $g(x) = 3x + 2$. Write an expression for:
   a) $f(g(x))$ b) $g(f(x))$
2. $f(x) = 2x - 5$ and $g(x) = x + 1$
   a) Evaluate $f(g(0))$.
   b) Evaluate $g(f(-2))$.
   c) Explain clearly why $f(g(x)) \neq g(f(x))$.
3. $f(x) = x^2 - 4$ and $g(x) = 2x + 1$
   a) Write an expression for $f(g(x))$.
   b) Solve the equation $f(g(x)) = 0$.
4. Find the inverse of these functions:
   a) $f(x) = \frac{5x + 2}{4}$ b) $h(x) = \frac{1}{2}x + 6$
5. Solve the following equations. Give your answers correct to 1 d.p.
   a) $2^{x-4} = 20$ b) $\frac{1}{2}\log_{10} x = \log_{10} 300 - 2$

TOPIC 4

Geometry

This topic will cover the following syllabus content:

4.4 Angles round a point
  Angles on a straight line and intersecting straight lines
  Vertically opposite angles
  Alternate and corresponding angles on parallel lines
  Angle sum of a triangle, quadrilateral and polygons
  Interior and exterior angles of a polygon
  Angles of regular polygons

4.5 Similarity
  Calculation of lengths of similar figures
  Area and volume scale factors

4.6 Theorem of Pythagoras and its converse in two and three dimensions
  Including:
    chord length and its distance of a chord from the centre of a circle
    distances on a grid

4.7 Vocabulary or circles
  Properties of circles:
    tangent perpendicular to radius at the point of contact
    tangents from a point
    angle in a semicircle
    angles at the centre and at the circumference on the same arc
    cyclic quadrilateral

Sections

SECTION 1 The Greeks

Many of the great Greek mathematicians came from the Greek Islands, from cities like Ephesus or Miletus (which are in present day Turkey) or from Alexandria in Egypt. This introduction briefly mentions some of the Greek mathematicians of 'The Golden Age'. You may wish to find out more about them.

Thales of Alexandria invented the 365 day calendar and predicted the dates of eclipses of the Sun and the moon.

Pythagoras of Samos founded a school of mathematicians and worked with geometry. His successor as leader was Theano, the first woman to hold a major role in mathematics.

Eudoxus of Asia Minor (Turkey) worked with irrational numbers like pi and discovered the formula for the volume of a cone.

Euclid of Alexandria formed what would now be called a university department. His book became the set text in schools and universities for 2000 years.

Apollonius of Perga (Turkey) worked on, and gave names to, the parabola, the hyperbola and the ellipse.

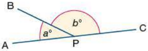
Archimedes is accepted today as the greatest mathematician of all time. However he was so far ahead of his time that his influence on his contemporaries was limited by their lack of understanding.

Archimedes (287–212BC)

SECTION 2 Angle properties

Angles are measures of turn. The most common measure of the size of an angle is the degree (°). There are many angle relationships in geometry, the most common of which are explained below.

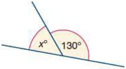
Angles on a straight line



The points APC lie on a straight line. A person standing at point P, initially facing point A, turns through an angle $a°$ to face point B and then turns a further angle $b°$ to face point C. The person has turned through half a complete turn and therefore rotated through 180°. Therefore $a° + b° = 180°$. This can be summarised as:

> Angles on a straight line, about a point, add up to 180°.

Worked example: Calculate the value of x in the diagram (left):

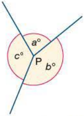


The sum of all the angles at a point on a straight line is 180°. Therefore:

$$\begin{aligned} x° + 130° &= 180° \\ x° &= 180° - 130° \end{aligned}$$

Therefore angle x is 50°

Angles at a point



The diagram (left) shows that if a person standing at P turns through each of the angles $a°$, $b°$ and $c°$ in turn, then the total amount he has rotated would be 360° (a complete turn)

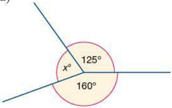
i.e. $a° + b° + c° = 360°$

> Angles about a point add up to 360°.

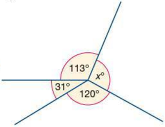
Exercise 4.1

1. Calculate the size of angle x in each of the following:

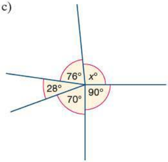
a)



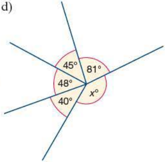
b)



c)

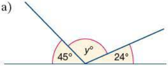


d)



2. Calculate the size of angle *y* in each of the following:

a)



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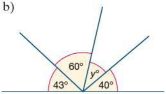
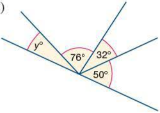


c)



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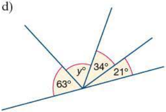
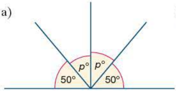
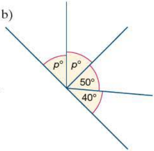


3. Calculate the size of angle *p* in each of the following:

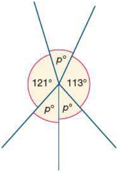
a)



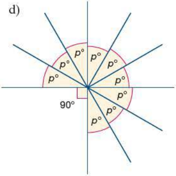
b)



c)



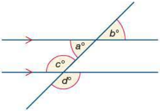
d)



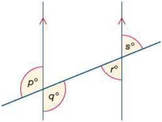
Angles formed within parallel lines

Exercise 4.2

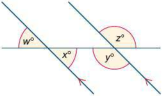
1. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



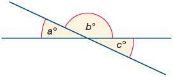
2. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



3. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



4. Write down what you have noticed about the angles you measured in Q1–3.



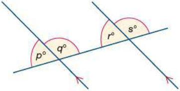
When two straight lines cross, it is found that the angles opposite each other are the same size. They are known as **vertically opposite angles**. By using the fact that angles at a point on a straight line add up to 180°, it can be shown why vertically opposite angles must always be equal in size.

$a° + b° = 180°$
$c° + b° = 180°$

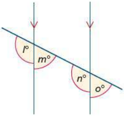
Therefore, a is equal to c.

Exercise 4.3

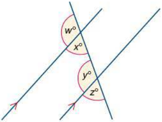
1. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



2. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.

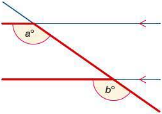


3. Draw a similar diagram to the one shown below. Measure carefully each of the labelled angles and write them down.



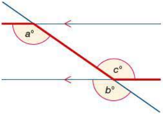
4. Write down what you have noticed about the angles you measured in Q1–3.

When a line intersects two parallel lines, as in the diagram below, it is found that certain angles are the same size.



The angles *a* and *b* are equal and are known as **corresponding angles**. Corresponding angles can be found by looking for an 'F' formation in a diagram.

A line intersecting two parallel lines also produces another pair of equal angles known as **alternate angles**. These can be shown to be equal by using the fact that both vertically opposite and corresponding angles are equal.



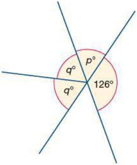
In the diagram above, $a = b$ (corresponding angles). But $b = c$ (vertically opposite). So $a = c$.

Angles *a* and *c* are alternate angles. These can be found by looking for a 'Z' formation in a diagram.

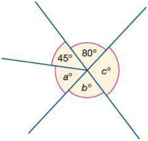
Exercise 4.4

In each of the following questions, some of the angles are given. Deduce, giving reasons, the size of the other labelled angles.

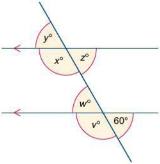
1.



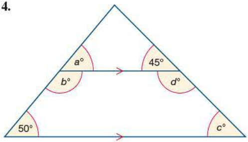
2.



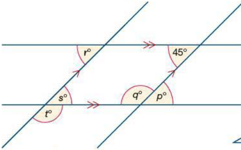
3.



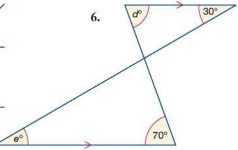
4.



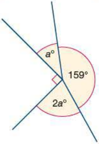
5.



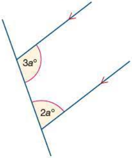
6.



7.

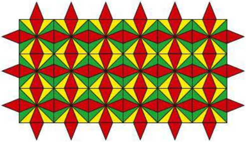


8.



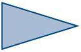
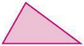
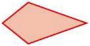
Properties of polygons

A polygon is a closed, two-dimensional shape formed by straight lines, with at least three sides. Examples of polygons include triangles, quadrilaterals and hexagons. The pattern below shows a number of different polygons **tessellating**; that is fitting together with no gaps or overlaps.



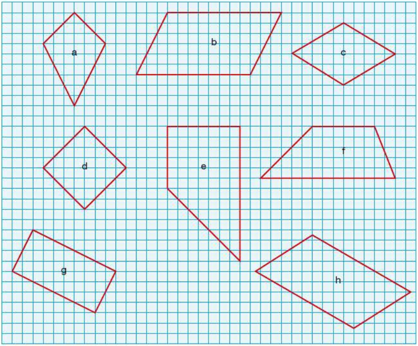
The polygons in the pattern on page 218 tessellate because they have particular properties.

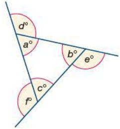
The properties of several triangles and quadrilaterals are listed below:

Name	Shape	Properties
Isosceles triangle		● Two sides are the same length ● Two angles are the same size
Equilateral triangle		● All sides are the same length ● All angles are the same size Note: An equilateral triangle is a special isosceles triangle.
Scalene triangle		● All sides are a different length ● All angles are a different size
Square		● All sides are the same length ● All interior angles are right angles ● Diagonals intersect at right angles
Rectangle		● Opposite sides are the same length ● All interior angles are right angles Note: A square is a special rectangle.
Rhombus		● All sides are of the same length ● Two pairs of parallel sides ● Opposite angles are equal ● Diagonals intersect at right angles
Parallelogram		● Opposite sides are the same length ● Opposite angles are equal ● Two pairs of parallel sides Note: A rhombus is a special parallelogram.
Trapezium		● One pair of parallel sides
Kite		● Two pairs of equal sides ● One pair of equal angles ● Diagonals intersect at right angles

Exercise 4.5

1. Identify as many different polygons as you can in the tessellating pattern on page 218.
2. Name each of the polygons drawn below. Give a reason for each answer.





Angle properties of polygons

In the triangle, the **interior angles** are labelled *a*, *b* and *c*, whilst the **exterior angles** are labelled *d*, *e* and *f*.

Imagine a person standing at one of the **vertices** (corners) and walking along the edges of the triangle until they at the start again. At each vertex they would have turned through an angle equivalent to the exterior angle at that point. This shows that, during the complete journey, they would have turned through an angle equivalent to one complete turn, i.e. 360°.

Therefore, $d^\circ + e^\circ + f^\circ = 360^\circ$.

It is also true that $a^\circ + d^\circ = 180^\circ$ (angles on a straight line) $b^\circ + e^\circ = 180^\circ$ and $c^\circ + f^\circ = 180^\circ$.

Therefore,
$$a^\circ + b^\circ + c^\circ + d^\circ + e^\circ + f^\circ = 540^\circ$$
$$a^\circ + b^\circ + c^\circ + 360^\circ = 540^\circ$$
$$a^\circ + b^\circ + c^\circ = 180^\circ$$

These findings lead us to two more important rules:

1. The exterior angles of a triangle (indeed of any polygon) add up to 360°.
2. The interior angles of a triangle add up to 180°.

By looking at the triangle again, it can now be stated that:

$$a° + d° = 180°.$$

and also $a° + b° + c° = 180°$

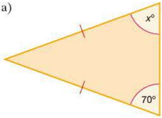
Therefore $d° = b° + c°$

The exterior angle of a triangle is equal to the sum of the opposite two interior angles.

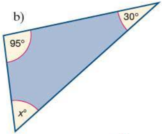
Exercise 4.6

1. For each of the triangles below, use the information given to calculate the size of angle ***x***:

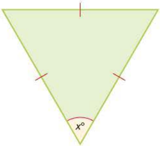
a)



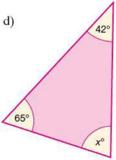
b)



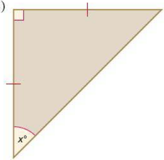
c)



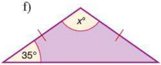
d)



e)

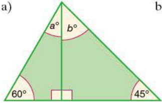


f)

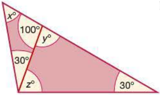


2. In each of the diagrams below, calculate the size of the labelled angles:

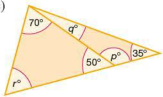
a)



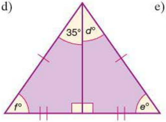
b)



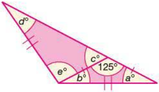
c)



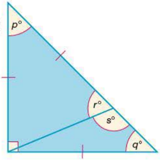
d)



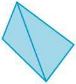
e)

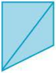


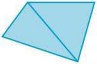
f)



In the quadrilaterals below, a straight line is drawn from one of the vertices to the opposite vertex. The result is to split the quadrilaterals into two triangles.





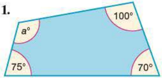


You already know that the sum of the angles in a triangle is 180°. Therefore, as a quadrilateral can be split into two triangles, the sum of the four angles of any quadrilateral must be 360°.

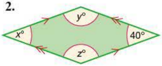
Exercise 4.7

In each of the diagrams below, calculate the size of the lettered angles.

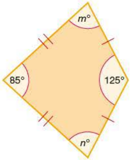
1.



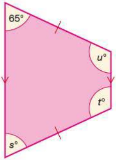
2.



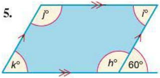
3.



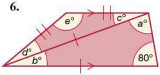
4.



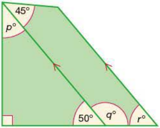
5.



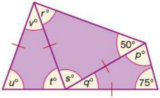
6.



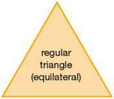
7.



8.



You already know that a polygon is a closed two-dimensional shape, bounded by straight lines. A **regular polygon** is distinctive in that all its sides are of equal length and all its angles of equal size. Below are some examples of regular polygons.

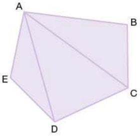


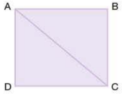
The name of each polygon is derived from the number of angles it contains. The following list identifies some of these polygons.

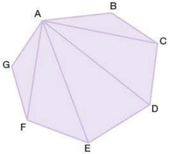
3 angles = **tri**angle
4 angles = **quad**rilateral (tetragon)
5 angles = **pent**agon
6 angles = **hex**agon
7 angles = **hept**agon (septagon)
8 angles = **oct**agon
9 angles = **non**agon
10 angles = **dec**agon
12 angles = **dodec**agon

The sum of the interior angles of a polygon

In the polygons below, a straight line is drawn from each vertex to vertex A.







(Note: the above shapes are **irregular** polygons since their sides are not of equal length.)

As can be seen, the number of triangles is always two less than the number of sides the polygon has, i.e. if there are n sides, then there will be $(n - 2)$ triangles.

Since the angles of a triangle add up to 180°, the sum of the interior angles of a polygon is therefore $180(n - 2)$ degrees.

Worked example

Find the sum of the interior angles of a regular pentagon and hence the size of each interior angle.

For a pentagon, $n = 5$.

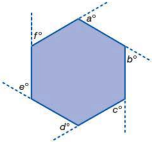
Therefore the sum of the interior angles $= 180(5 - 2)^\circ$

$$= 180 \times 3$$

$$= 540^\circ$$

For a regular pentagon the interior angles are of equal size.

Therefore each angle is $\frac{540}{5} = 108^\circ$.



The sum of the exterior angles of a polygon

The angles marked *a*, *b*, *c*, *d*, *e* and *f*, represent the exterior angles of the regular hexagon drawn. As we have already found, the sum of the exterior angles of any polygon is 360°.

If the polygon is regular and has *n* sides, then each exterior angle is $\frac{360^\circ}{n}$.

Worked examples

a) Find the size of an exterior angle of a regular nonagon.

$$\frac{360^\circ}{9} = 40^\circ$$

b) Calculate the number of sides a regular polygon has if each exterior angle is 15°.

$$n = \frac{360^\circ}{15}$$
$$= 24$$

The polygon has 24 sides.

Exercise 4.8

1. Find the sum of the interior angles of the following polygons:
a) a hexagon b) a nonagon c) a heptagon

2. Find the value of each interior angle of the following regular polygons:
a) an octagon b) a square
c) a decagon d) a dodecagon

3. Find the size of each exterior angle of the following regular polygons:
a) a pentagon b) a dodecagon c) a heptagon

4. The exterior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a) 20° b) 36° c) 10°
d) 45° e) 18° f) 3°

5. The interior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a) 108° b) 150° c) 162°
d) 156° e) 171° f) 179°

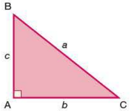
6. Calculate the number of sides a regular polygon has if an interior angle is five times the size of an exterior angle.

7. Copy and complete the table below for regular polygons:

Number of sides	Name	Sum of exterior angles	Size of an exterior angle	Sum of interior angles	Size of an interior angle
3					
4					
5					
6					
7					
8					
9					
10					
12					

SECTION 3

Pythagoras' theorem



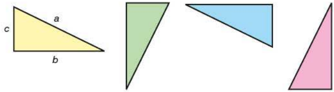
Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle.

Pythagoras' theorem states that:

$$a^2 = b^2 + c^2$$

There are many proofs of Pythagoras' theorem. One of them is shown below.

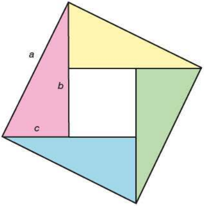
Consider four congruent (identical) right-angled triangles:



Three are rotations of the first triangle by 90°, 180° and 270° clockwise respectively.

Each triangle has an area equal to $\frac{bc}{2}$.

The triangles can be joined together to form a square of side length a as shown:



The square has a hole at its centre of side length b – c.

Therefore, the area of centre square is $(b-c)^2 = b^2 -2bc + c^2$

the area of the four triangles is $4 \times \frac{bc}{2} = 2bc$

the area of the large square is a^2

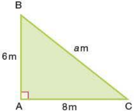
Therefore

$$a^2 = (b-c)^2 + 2bc$$
$$a^2 = b^2 - 2bc + c^2 + 2bc$$
$$a^2 = b^2 + c^2$$

hence proving Pythagoras' theorem.

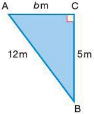
Worked examples

a) Calculate the length of the side BC.
Using Pythagoras:



$$a^2 = b^2 + c^2$$
$$a^2 = 8^2 + 6^2$$
$$a^2 = 64 + 36 = 100$$
$$a = \sqrt{100}$$
$$a = 10$$

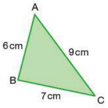
BC = 10 m



b) Calculate the length of the side AC.
Using Pythagoras:

$$a^2 = b^2 + c^2$$
$$a^2 - c^2 = b^2$$
$$b^2 = 144 - 25 = 119$$
$$b = \sqrt{119}$$
$$b = 10.9 \text{ (3 s.f.)}$$

AC = 10.9 m (3 s.f.)



The converse of Pythagoras' theorem can also be used to show whether or not a triangle is right-angled.

In the triangle ABC to the left, the lengths of the three sides are given, but it is not indicated whether any of the angles are a right angle. It is important not to assume that a triangle is right-angled just because it may look like it is.

If triangle ABC is right-angled, then it will satisfy Pythagoras' theorem.

i.e. $AC^2 = AB^2 + BC^2$

$$9^2 = 6^2 + 7^2$$

$$81 = 36 + 49$$

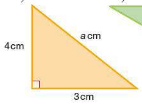
$$81 = 85$$

This is clearly incorrect, therefore triangle ABC is not right-angled.

Exercise 4.9

In each of the diagrams in Q.1 and 2, use Pythagoras' theorem to calculate the length of the marked side.

1. a)



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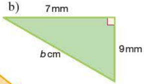
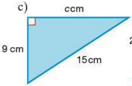
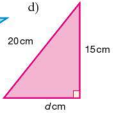


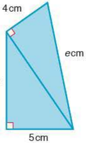
c)



d)



2. a)



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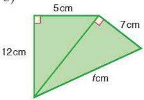

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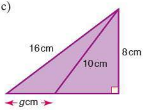

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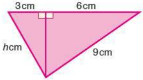
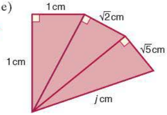
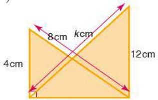


e)

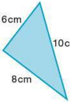


f)

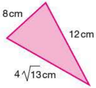


3. By applying Pythagoras' theorem, decide which of the following triangles are right-angled.

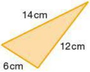
a)



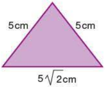
b)



c)



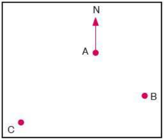
d)



4. Villages A, B and C lie on the edge of the Namib desert. Village A is 30 km due North of village C. Village B is 65 km due East of A.

Calculate the shortest distance between villages C and B, giving your answer to the nearest 0.1 km.

5. Town X is 54 km due West of town Y. The shortest distance between town Y and town Z is 86 km. If town Z is due South of X calculate the distance between X and Z, giving your answer to the nearest kilometre.

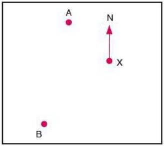


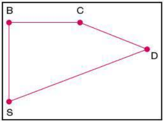
6. Village B is on a bearing of 135° and at a distance of 40 km from village A. Village C is on a bearing of 225° and a distance of 62 km from village A.

a) Show that triangle ABC is right-angled.
b) Calculate the distance from B to C, giving your answer to the nearest 0.1 km.

7. Two boats set off from X at the same time. Boat A sets off on a bearing of 325° and with a velocity of 14km/h. Boat B sets off on a bearing of 235° with a velocity of 18km/h.

Calculate the distance between the boats after they have been travelling for 2.5 hours. Give your answer to the nearest metre.

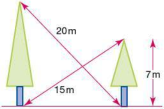




8. A boat sets off on a trip from S. It heads towards B, a point 6km away and due North. At B it changes direction and heads towards point C, also 6km away and due East of B. At C it changes direction once again and heads on a bearing of 135° towards D which is 13km from C.

a) Calculate the distance between S and C to the nearest 0.1km.

b) Calculate the distance the boat will have to travel if it is to return to S from D.



9. Two trees are standing on flat ground.

The height of the smaller tree is 7m. The distance between the top of the smaller tree and the base of the taller tree is 15m.

The distance between the top of the taller tree and the base of the smaller tree is 20m.

a) Calculate the horizontal distance between the two trees.

b) Calculate the height of the taller tree.

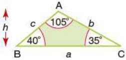
Pythagoras' theorem can also be applied to problems in three dimensions. This is covered in Topic 8.

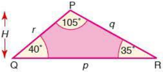
SECTION 4 Similarity

Similar shapes

Two polygons are said to be **similar** if a) they are equi-angular and b) corresponding sides are in proportion.

For triangles, being equi-angular implies that corresponding sides are in proportion. The converse is also true.





In the diagrams (left) Δ ABC and Δ PQR are similar.

For similar figures the ratios of the lengths of the sides are the same and represent the **scale factor**, i.e.

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k \text{ (where } k \text{ is the scale factor of enlargement)}$$

The heights of similar triangles are proportional also:

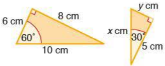
$$\frac{H}{h} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k$$

The ratio of the areas of similar triangles (the **area factor**) is equal to the square of the scale factor.

$$\frac{\text{Area of } \Delta pqr}{\text{Area of } \Delta \text{ABC}} = \frac{\frac{1}{2}H \times p}{\frac{1}{2}h \times a} = \frac{H}{h} \times \frac{p}{a} = k \times k = k^2$$

Exercise 4.10

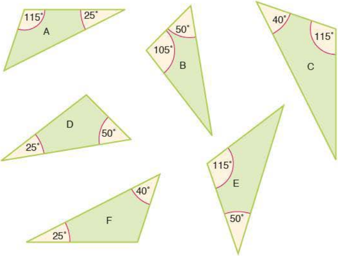
1. a) Explain why the two triangles below are similar.



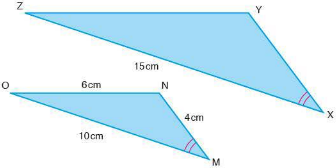
b) Calculate the scale factor which reduces the larger triangle to the smaller one.

c) Calculate the value of x and the value of y.

2. Which of the triangles below are similar?

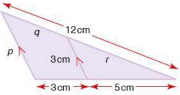


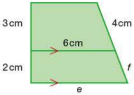
3. The triangles below are similar.



a) Calculate the length XY.
b) Calculate the length YZ.

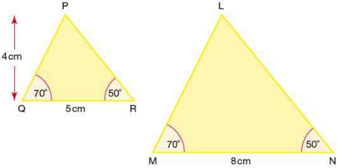
4. In the triangle to the right calculate the lengths of sides *p*, *q* and *r*.





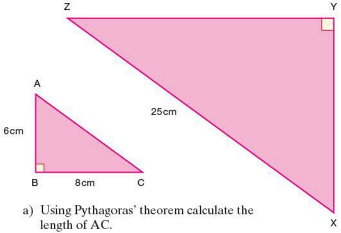
5. In the trapezium to the left the lengths of sides *e* and *f*.

6. The triangles PQR and LMN are similar.

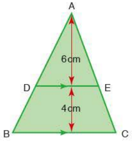


Calculate:
a) the area of Δ PQR
b) the scale factor of enlargement
c) the area of Δ LMN.

7. The triangles ABC and XYZ below are similar.

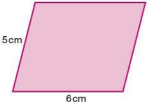


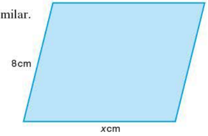
a) Using Pythagoras' theorem calculate the length of AC.
b) Calculate the scale factor of enlargement.
c) Calculate the area of Δ XYZ.



8. The triangle ADE shown (left) has an area of 12cm^2.
a) Calculate the area of Δ ABC.
b) Calculate the length BC.

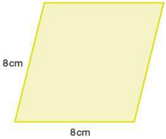
9. The following parallelograms are similar.

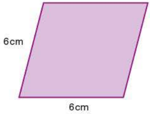




Calculate the length of the side marked *x*.

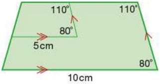
10. The diagram below shows two rhombuses.





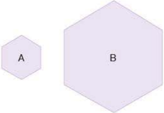
Explain, giving reasons, whether the two rhombuses are definitely similar.

11. The diagram to the right shows a trapezium within a trapezium. Explain, giving reasons, whether the two trapezia are definitely similar.



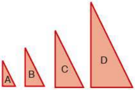
Exercise 4.11

1. In the hexagons shown, hexagon B is an enlargement of hexagon A by a scale factor of 2.5.



If the area of A is 8cm^2, calculate the area of B.

2. P and Q are two regular pentagons. Q is an enlargement of P by a scale factor of 3. If the area of pentagon Q is 90cm^2, calculate the area of P.



3. On the left is a row of four triangles A, B, C and D. Each is an enlargement of the previous one by a scale factor of 1.5
- a) If the area of C is 202.5cm^2, calculate the area of:
  - i) triangle D   ii) triangle B   iii) triangle A.
- b) If the triangles were to continue in this sequence, which letter triangle would be the first to have an area greater than 15000cm^2?

4. A square is enlarged by increasing the length of its sides by 10%. If the length of its sides was originally 6 cm, calculate the area of the enlarged square.

5. A square of side length 4 cm is enlarged by increasing the lengths of its sides by 25% and then increasing them by a further 50%. Calculate the area of the final square.

6. An equilateral triangle has an area of 25cm^2. If the lengths of its sides are reduced by 15%, calculate the area of the reduced triangle.

Area and volume of similar shapes

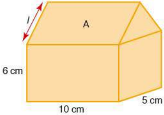
Earlier in the topic we found the following relationship between the scale factor and the area factor of enlargement:

Area factor = (scale factor)^2

A similar relationship can be stated for volumes of similar shapes:

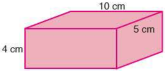
i.e. Volume factor = $(\text{scale factor})^3$

Exercise 4.12



1. The diagram on the left is a scale model of a garage. Its width is 5 cm, its length 10 cm and the height of its walls 6 cm.
   a) If the width of the real garage is 4 m, calculate:
      i) the length of the real garage
      ii) the real height of the garage wall.
   b) If the apex of the roof of the real garage is 2 m above the top of the walls, use Pythagoras' theorem to find the real slant length *l*.
   c) What is the actual area of the roof section marked A?

2. A cuboid has dimensions as shown:



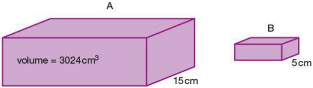
If the cuboid is enlarged by a scale factor of 2.5, calculate:

   a) the total surface area of the original cuboid
   b) the total surface area of the enlarged cuboid
   c) the volume of the original cuboid
   d) the volume of the enlarged cuboid.

3. A cube has side length 3 cm.
   a) Calculate its total surface area.
   b) If the cube is enlarged and has a total surface area of 486 cm^2, calculate the scale factor of enlargement.
   c) Calculate the volume of the enlarged cube.

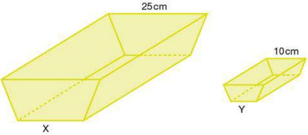
4. Two cubes P and Q are of different sizes. If n is the ratio of their corresponding sides, express in terms of n:
   a) the ratio of their surface areas
   b) the ratio of their volumes.

5. The cuboids A and B shown below are similar.



Calculate the volume of cuboid B.

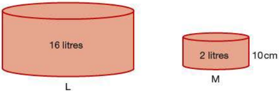
6. Two similar troughs X and Y are shown below.



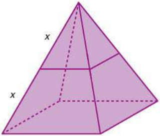
If the capacity of X is 10 litres, calculate the capacity of Y.

Exercise 4.13

1. The two cylinders L and M shown below are similar.

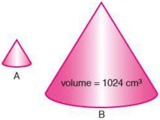


If the height of cylinder M is 10 cm, calculate the height of cylinder L.

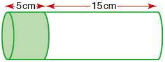


2. A square-based pyramid (left) is cut into two shapes by a cut running parallel to the base and made half-way up.
   a) Calculate the ratio of the volume of the smaller pyramid to that of the original one.
   b) Calculate the ratio of the volume of the small pyramid to that of the truncated base.

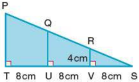
3. The two cones A and B shown below are similar. Cone B is an enlargement of A by a scale factor of 4.



If the volume of cone B is 1024 cm^3, calculate the volume of cone A.

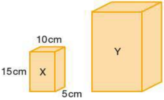


4. a) Stating your reasons clearly, decide whether the two cylinders shown to the left are similar or not.
   b) What is the ratio of the curved surface area of the shaded cylinder to that of the unshaded cylinder?



5. The diagram (left) shows a triangle.
   a) Calculate the area of Δ RSV.
   b) Calculate the area of Δ QSU.
   c) Calculate the area of Δ PST.

6. The area of an island on a map is $30\,\text{cm}^2$. The scale used on the map is 1:100 000.
   a) Calculate the area in square kilometres of the real island.
   b) An airport on the island is on a rectangular piece of land measuring 3 km by 2 km. Calculate the area of the airport on the map in cm^2.



7. The two packs of cheese X and Y (left) are similar.

   The total surface area of pack Y is four times that of pack X.

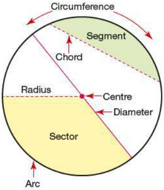
   Calculate:

   a) the dimensions of pack Y
   b) the mass of pack X if pack Y has a mass of 800 g.

SECTION 5

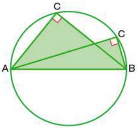
Properties of circles

You will already be familiar with the terms used to describe apects of the circle shown in the diagram.



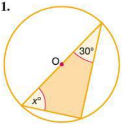
The angle in a semi-circle

If AB represents the diameter of the circle, then the angle at C is 90°.

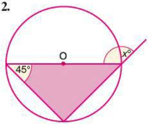


Exercise 4.14 In each of the following diagrams, O marks the centre of the circle. Calculate the value of *x* in each case.

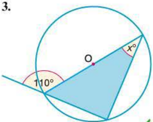
1.



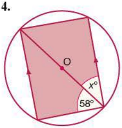
2.



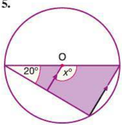
3.



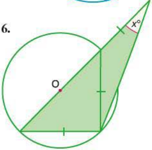
4.



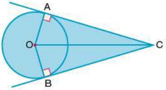
5.



6.



The angle between a tangent and a radius of a circle



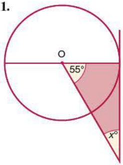
The angle between a tangent at a point and the radius to the same point on the circle is a right angle.

Triangles OAC and OBC are congruent as ∠OAC and ∠OBC are right angles, OA = OB because they are both radii and OC is common to both triangles.

Exercise 4.15

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

1.



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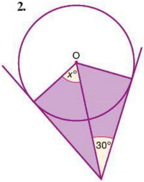

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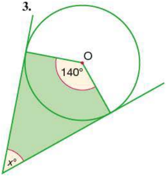

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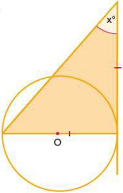

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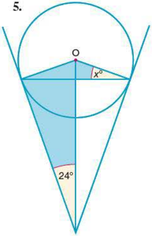

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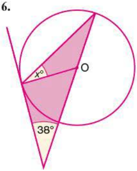

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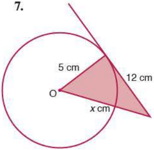

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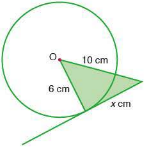

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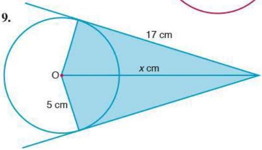
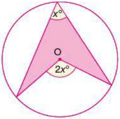
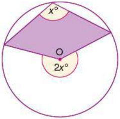


Angle at the centre of a circle

The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc.

Both diagrams below illustrate this theorem.



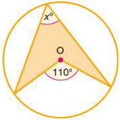


A proof for this theorem is given in a Personal Tutor on the HodderPlus website.

Exercise 4.16

In each of the following diagrams, O marks the centre of the circle. Calculate the size of the lettered angles:

1.



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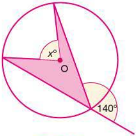

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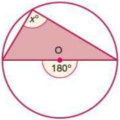

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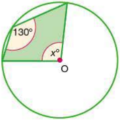

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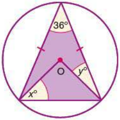

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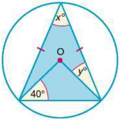
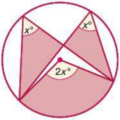




Angles in the same segment

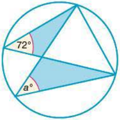
Angles in the same segment of a circle are equal.

This can be explained simply by using the theorem that the angle subtended at the centre is twice the angle on the circumference. Looking at the diagram (left), if the angle at the centre is $2x°$, then each of the angles at the circumference must be equal to $x°$.

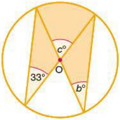
Exercise 4.17

Calculate the lettered angles in the following diagrams:

1.



2.



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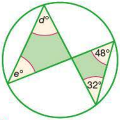

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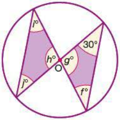

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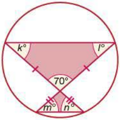

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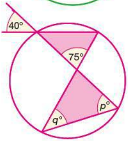
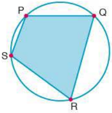
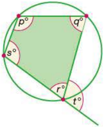


Angles in opposite segments





Points P, Q, R and S all lie on the circumference of the circle above. They are called **concyclic points**. Joining the points P, Q, R and S produces a cyclic quadrilateral.

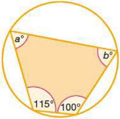
The opposite angles are **supplementary**, i.e. they add up to 180°. A proof for this is given in a Personal Tutor on the HodderPlus website.

Since $p^\circ + r^\circ = 180^\circ$ (supplementary angles) and $r^\circ + t^\circ = 180^\circ$ (angles on a straight line), it follows that $p^\circ = t^\circ$.

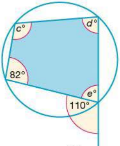
Therefore the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Exercise 4.18 Calculate the size of the lettered angles in each of the following:

1.



2.



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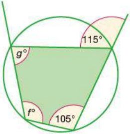
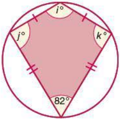


4.



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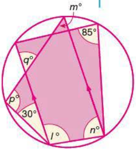

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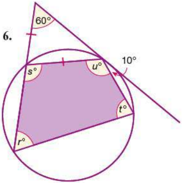
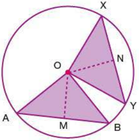


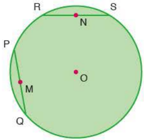


Equal chords and perpendicular bisectors

If chords AB and XY are of equal length, then, since OA, OB, OX and OY are radii, the triangles OAB and OXY are congruent isosceles triangles. It follows that:

- the section of a line of symmetry OM through Δ OAB is the same length as the section of a line of symmetry ON through Δ OXY
- OM and ON are perpendicular bisectors of AB and XY respectively.

Exercise 4.19



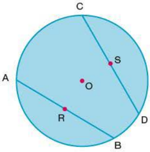
1. In the diagram (left) O is the centre of the circle, PQ and RS are chords of equal length and M and N are their respective midpoints.

a) What kind of triangle is Δ POQ?

b) Describe the line ON in relation to RS.

c) If ∠POQ is 80°, calculate ∠OQP.

d) Calculate ∠ORS.



2. In the diagram (left) O is the centre of the circle. AB and CD are equal chords and the points R and S are their midpoints respectively.

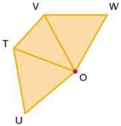
Identify which of these statements are true and which are false, giving reasons for your answers.

a) ∠COD = 2 × ∠AOR

b) OR = OS

c) If ∠ROB is 60°, then Δ AOB is equilateral.

d) OR and OS are perpendicular bisectors of AB and CD respectively.



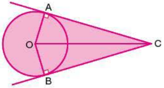
3. Using the diagram (left) identify which of the following statements are true and which are false, giving reasons for your answers.

a) If Δ VOW and Δ TOU are isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.

b) If Δ VOW and Δ TOU are congruent isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.

Tangents from an external point

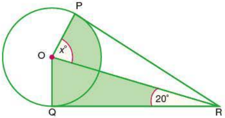
Triangles OAC and OBC are congruent since ∠OAC and ∠OBC are right angles, OA = OB because they are both radii, and OC is common to both triangles. Hence AC = BC.
In general, therefore, tangents being drawn to the same circle from an external point are equal in length.



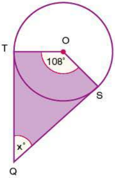
Exercise 4.20

1. Copy each of the diagrams below and calculate the size of the angle marked $x°$ in each case. Assume that the lines drawn from points on the circumference are tangents.

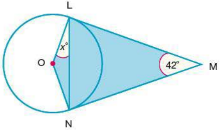
a)



b)

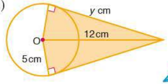


c)

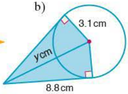


2. Copy each of the diagrams below and calculate the length of the side marked y cm in each case. Assume that the lines drawn from points on the circumference are tangents.

a)

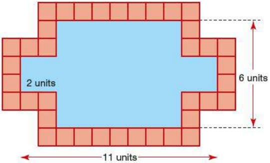


b)



Fountain borders

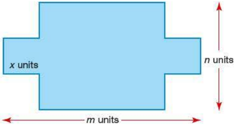
The Alhambra Palace in Granada, Spain has many fountains which pour water into pools. Many of the pools are surrounded by beautiful ceramic tiles. This investigation looks at the number of square tiles needed to surround a particular shape of pool.



The diagram above shows a rectangular pool 11 × 6 units, in which a square of dimension 2 × 2 units is taken from each corner.

The total number of unit square tiles needed to surround the pool is 38.

The shape of the pools can be generalised as shown below:

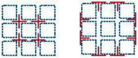


1. Investigate the number of unit square tiles needed for different sized pools. Record your results in an ordered table.
2. From your results write an algebraic rule in terms of m, n and x (if necessary) for the number of tiles T needed to surround a pool.
3. Justify, in words and using diagrams, why your rule works.

Tiled walls

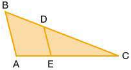
Many cultures have used tiles to decorate buildings. Putting tiles on a wall takes skill. These days, to make sure that each tile is in the correct position 'spacers' are used between the tiles.

You can see from the diagram that there are + shaped and T shaped spacers.



1. Draw other sized squares and rectangles, and investigate the relationship between the dimensions of the shape (length and width) and the number of + shaped and T shaped spacers.
2. Record your results in an ordered table.
3. Write an algebraic rule for the number of + shaped spacers c in a rectangle l tiles long by w tiles wide.
4. Write an algebraic rule for the number of T shaped spacers t in a rectangle l tiles long by w tiles wide.

ICT Activity 1



In this activity you will be using a dynamic geometry package such as Cabri or Geogebra to demonstrate that for the triangle left

$$\frac{AB}{ED} = \frac{AC}{EC} = \frac{BC}{DC}$$

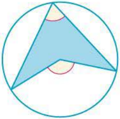
1. a) Using the geometry package construct the triangle ABC.
   b) Construct the line segment ED such that it is parallel to AB. (You will need to construct a line parallel to AB first and then attach the line segment ED to it.)
   c) Using a 'measurement' tool, measure each of the lengths AB, AC, BC, ED, EC and DC.
   d) Using a 'calculator' tool, calculate the ratio $\frac{AB}{ED}$, $\frac{AC}{EC}$, $\frac{BC}{DC}$
2. Comment on your answers to Q.1(d).
3. a) Grab vertex B and move it to a new position. What happens to the ratios you calculated in Q.1(d)?
   b) Grab the vertices A and C in turn and move them to new positions. What happens to the ratios? Explain why this happens.
4. Grab point D and move it to a new position along the side BC. Explain, giving reasons, what happens to the ratios.

ICT Activity 2

Using a geometry package, such as Cabri or Geogebra, demonstrate the following angle properties of a circle:

i) The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc.

The diagram below demonstrates the construction that needs to be formed:

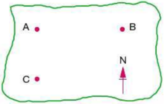


ii) The angles in the same segment of a circle are equal.

iii) The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

SECTION 7 Student assessments

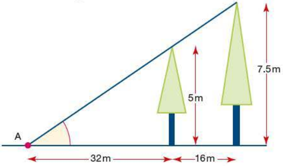
Student assessment 1

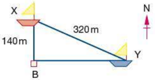


1. A map shows three towns A, B and C. Town A is due North of C. Town B is due East of A.

The distance AC is 75km and the distance of AB is 100km. Calculate the distance between towns B and C.

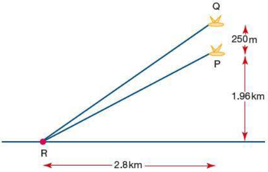
2. Calculate the distance from A to the top of each of the two trees in the diagram below.





3. Two boats X and Y, sailing in a race, are shown in the diagram (left). Boat X is 140 m due North of a buoy B. Boat Y is due East of buoy B. Boats X and Y are 320 m apart. Calculate the distance BY.

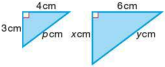
4. Two hawks P and Q are flying vertically above one another. Hawk Q is 250 m above hawk P. They both spot a snake at R. The height of P above the ground is 1.96 km.



Using the information given, calculate:

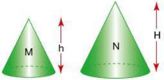
a) the distance between P and R

b) the distance between Q and R.



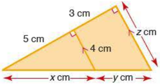
Student assessment 2

1. The two triangles shown (left) are similar.
   a) Using Pythagoras' theorem, calculate the value of p.
   b) Calculate the values of x and y.

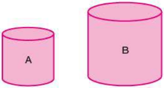


2. Cones M and N are similar.
   a) Express the ratio of their surface areas in the form, area of M: area of N.
   b) Express the ratio of their volumes in the form, volume of M: volume of N.

3. Calculate the values of x, y and z in the triangle below.



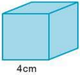
4. The tins A and B shown below are similar. The capacity of tin B is three times that of tin A. If the label on tin A has an area of 75 cm², calculate the area of the label on tin B.



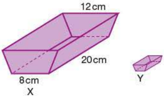
5. The cube shown on the right is enlarged by a scale factor of 2.5.

a) Calculate the volume of the enlarged cube.

b) Calculate the surface area of the enlarged cube.



6. The two troughs X and Y shown below are similar.



The scale factor of enlargement from Y to X is 4. If the capacity of trough X is 1200 cm³, calculate the capacity of trough Y.

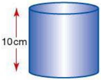
7. The rectangular floor plan of a house measures 8 cm by 6 cm. If the scale of the plan is 1 : 50, calculate:

a) the dimensions of the actual floor

b) the area of the actual floor in m².

8. The volume of the cylinder shown on the right is 400 cm³.

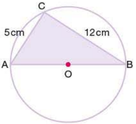
Calculate the volume of a similar cylinder formed by enlarging the one shown by a scale factor 2.



Student assessment 3

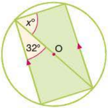
NB: Diagrams are not drawn to scale.

1. If AB is the diameter of the circle AC = 5 cm and BC = 12 cm, calculate:
   a) the size of angle ACB
   b) the length of the radius of the circle.

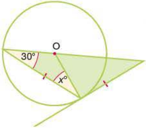


In Q.2–5, O marks the centre of the circle. Calculate the size of the angle marked ***x*** in each case.

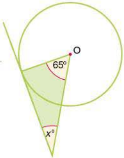
2.



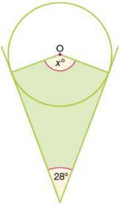
3.

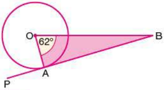


4.



5.

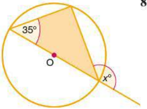




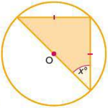
6. If OA is a radius of the circle and PB the tangent to the circle at A, calculate angle ABO.

In Q.7–10, O marks the centre of the circle. Calculate the size of the angle marked *x* in each case.

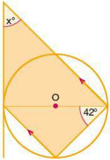
7.



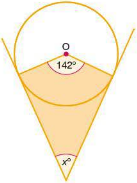
8.



9.



10.



Student assessment 4

NB: Diagrams are not drawn to scale.

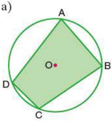
1. In the following diagrams, O is the centre of the circle. Identify which angles are:

i) supplementary angles

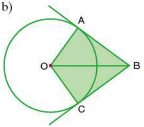
ii) right angles

iii) equal.

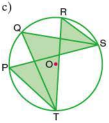
a)



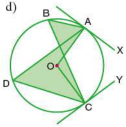
b)



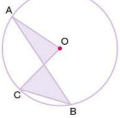
c)



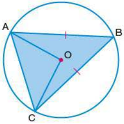
d)



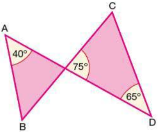
2. If ∠AOC is 72°, calculate ∠ABC.



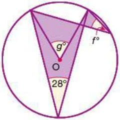
3. If ∠AOB = 130°, calculate ∠ABC, ∠OAB and ∠CAO.



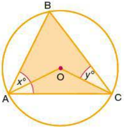
4. Show that ABCD is a cyclic quadrilateral.



5. Calculate *f* and *g*.

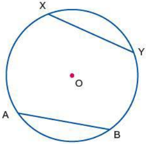


6. If $y = 22.5$, calculate the value of x.

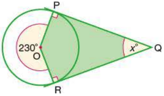


Student assessment 5

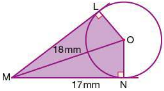
1. If O is the centre of the circle and the lengths AB and XY are equal, prove that Δ AOB and Δ XOY are congruent.



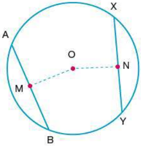
2. Given that PQ and QR are both tangents to the circle, calculate the size of the angle marked $x°$.



3. Calculate the diameter of the circle given that LM and MN are both tangents to the circle, O is its centre and OM = 18 mm.

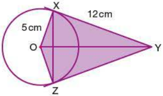


4. In the diagram, OM and ON are perpendicular bisectors of AB and XY respectively. OM = ON.



Prove that AB and XY are chords of equal length.

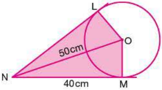
5. In the diagram, XY and YZ are both tangents to the circle with centre O. Calculate the length OY.



6. In the diagram, LN and MN are both tangents to the circle centre O. NM = 40cm and ON = 50cm.

a) Calculate the radius of the circle.

b) Calculate the circumference of the circle.



TOPIC 5 Transformations and vectors

This topic will cover the following syllabus content:

5.1 Notation:
Vector **a**; directed line segment $\overrightarrow{AB}$; component form $\begin{pmatrix} x \\ y \end{pmatrix}$

5.2 Addition of vectors using directed line segments or number pairs
Negative of a vector, subtraction of vectors
Multiplication of a vector by a scalar

5.3 Magnitude $|\mathbf{a}|$

5.4 Transformations on the cartesian plane:
translation, reflection, rotation, enlargement, (reduction), stretch
Description of a translation using the Notation in 5.1

5.5 Inverse of a transformation

5.6 Combined transformations

Sections

SECTION 1 The Italians

Fibonacci (1170–1250)

Leonardo Pisano (known today as Fibonacci) introduced new methods of arithmetic to Europe, from the Hindus, Persians and Arabs. He discovered the sequence 1, 1, 2, 3, 5, 8, 13, … which is now called the Fibonacci sequence, and some of its occurrences in nature. He also brought the decimal system, algebra and the 'lattice' method of multiplication to Europe. Fibonacci has been called the 'most talented mathematician of the middle ages'. Many books say that he brought Islamic mathematics to Europe, but in Fibonacci's own introduction to *Liber Abaci*, he credits the Hindus.

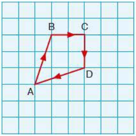
The Renaissance began in Italy. Art, architecture, music and the sciences flourished. However the Roman Catholic Church was both powerful and resistant to change.

Girolamo Cardano (1501–1576) wrote his great mathematical book *Ars Magna* (Great Art) in which he showed, among much algebra that was new, calculations involving the solutions to cubic equations. He wrote this book, the first algebra book in Latin, to great acclaim. He was charged with heresy in 1570 because the church did not approve of his work on astrology. Although he was found innocent and continued to study mathematics, no other work of his was ever published.

SECTION 2 Simple vectors

A **translation** (a sliding movement) can be described using column vectors. A column vector describes the movement of the object in both the x direction and the y direction.

Worked example



i) Describe the translation from A to B in the diagram (left) in terms of a column vector.

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. 1 unit in the *x* direction, 3 units in the *y* direction

ii) Describe $\overrightarrow{BC}$ in terms of a column vector.

$\overrightarrow{BC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

iii) Describe $\overrightarrow{CD}$ in terms of a column vector.

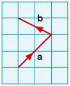
$\overrightarrow{CD} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

iv) Describe $\overrightarrow{DA}$ in terms of a column vector.

$\overrightarrow{DA} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Translations can also be named by a single letter. The direction of the arrow indicates the direction of the translation.

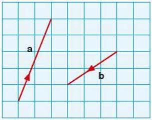
Worked example



Define **a** and **b** in the diagram using column vectors.

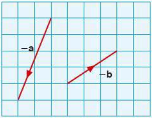
$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Note: When you represent vectors by single letters, for example **a**, in handwritten work, you should write them as <u>a</u>.



If $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$, they can be represented diagrammatically as shown (left).

The diagrammatic representation of –**a** and –**b** is shown below.

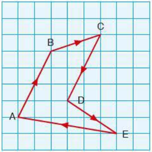


It can be seen from the diagram above that $-\mathbf{a} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and $-\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

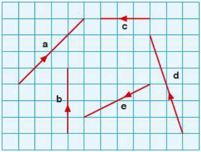
Exercise 5.1

In Q.1 and 2 describe each translation using a column vector.

1. a) $\overrightarrow{AB}$
   b) $\overrightarrow{BC}$
   c) $\overrightarrow{CD}$
   d) $\overrightarrow{DE}$
   e) $\overrightarrow{EA}$
   f) $\overrightarrow{AE}$
   g) $\overrightarrow{DA}$
   h) $\overrightarrow{CA}$
   i) $\overrightarrow{DB}$

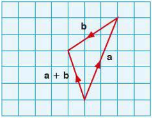


2. a) **a**
   b) **b**
   c) **c**
   d) **d**
   e) **e**
   f) –**b**
   g) –**c**
   h) –**d**
   i) –**a**



3. Draw and label the following vectors on a square grid:

   a) $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   b) $\mathbf{b} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$
   c) $\mathbf{c} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$
   d) $\mathbf{d} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$
   e) $\mathbf{e} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$
   f) $\mathbf{f} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$
   g) –**c**
   h) –**b**
   i) –**f**



Addition and subtraction of vectors

Vectors can be added together and represented diagrammatically as shown (left).

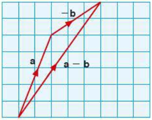
The translation represented by **a** followed by **b** can be written as a single transformation **a** + **b**:

i.e. $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Worked example

$\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

i) Draw a diagram to represent **a** – **b**, where **a** – **b** = (**a**) + (–**b**).



ii) Calculate the vector represented by $\mathbf{a} - \mathbf{b}$.

$$\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-3\\-2\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix}$$

Exercise 5.2

In the following questions,

$$\mathbf{a} = \begin{pmatrix}3\\4\end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix}-2\\1\end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix}-4\\-3\end{pmatrix} \qquad \mathbf{d} = \begin{pmatrix}3\\-2\end{pmatrix}$$

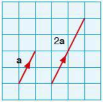
1. Draw vector diagrams to represent the following:
a) $\mathbf{a} + \mathbf{b}$ b) $\mathbf{b} + \mathbf{a}$ c) $\mathbf{a} + \mathbf{d}$
d) $\mathbf{d} + \mathbf{a}$ e) $\mathbf{b} + \mathbf{c}$ f) $\mathbf{c} + \mathbf{b}$

2. What conclusions can you draw from your answers to Q.1?

3. Draw vector diagrams to represent the following:
a) $\mathbf{b} - \mathbf{c}$ b) $\mathbf{d} - \mathbf{a}$ c) $-\mathbf{a} - \mathbf{c}$
d) $\mathbf{a} + \mathbf{c} - \mathbf{b}$ e) $\mathbf{d} - \mathbf{c} - \mathbf{b}$ f) $-\mathbf{c} + \mathbf{b} + \mathbf{d}$

4. Represent each of the vectors in Q.3 by a single column vector.

Multiplying a vector by a scalar

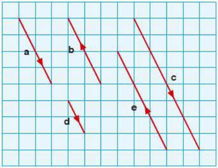


Look at the two vectors in the diagram (left).

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \qquad 2\mathbf{a} = 2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$$

Worked example

If $\mathbf{a} = \begin{pmatrix}2\\-4\end{pmatrix}$ express the vectors **b**, **c**, **d** and **e** in terms of **a**.

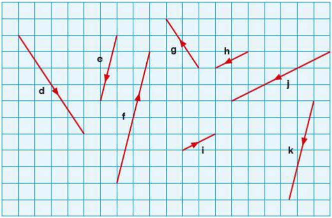


$\mathbf{b} = -\mathbf{a}$ $\mathbf{c} = 2\mathbf{a}$ $\mathbf{d} = \frac{1}{2}\mathbf{a}$ $\mathbf{e} = -\frac{3}{2}\mathbf{a}$

Exercise 5.3

1. $\mathbf{a} = \begin{pmatrix}1\\4\end{pmatrix}$ $\mathbf{b} = \begin{pmatrix}-4\\-2\end{pmatrix}$ $\mathbf{c} = \begin{pmatrix}-4\\6\end{pmatrix}$

Express the following vectors in terms of either **a**, **b** or **c**.



2. $\mathbf{a} = \begin{pmatrix}2\\3\end{pmatrix}$ $\mathbf{b} = \begin{pmatrix}-4\\-1\end{pmatrix}$ $\mathbf{c} = \begin{pmatrix}-2\\4\end{pmatrix}$

Represent each of the following as a single column vector:

a) 2**a** b) 3**b** c) –**c**

d) **a** + **b** e) **b** – **c** f) 3**c** – **a**

g) 2**b** – **a** h) $\frac{1}{2}(\mathbf{a} - \mathbf{b})$ i) 2**a** – 3**c**

3. $\mathbf{a} = \begin{pmatrix}-2\\3\end{pmatrix}$ $\mathbf{b} = \begin{pmatrix}0\\-3\end{pmatrix}$ $\mathbf{c} = \begin{pmatrix}4\\-1\end{pmatrix}$

Express each of the following vectors in terms of **a**, **b** and **c**:

a) $\begin{pmatrix}-4\\6\end{pmatrix}$ b) $\begin{pmatrix}0\\3\end{pmatrix}$ c) $\begin{pmatrix}4\\-4\end{pmatrix}$

d) $\begin{pmatrix}-2\\6\end{pmatrix}$ e) $\begin{pmatrix}8\\-2\end{pmatrix}$ f) $\begin{pmatrix}10\\-5\end{pmatrix}$

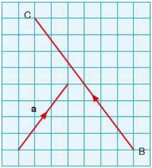
SECTION 3

Magnitude of a vector

The **magnitude** or size of a vector is represented by its length, i.e. the longer the length, the greater the magnitude. The magnitude of a vector **a** or $\overrightarrow{AB}$ is denoted by $|\mathbf{a}|$ or $|\overrightarrow{AB}|$ respectively and is calculated using Pythagoras' theorem.

Worked examples

$\mathbf{a} = \begin{pmatrix}3\\4\end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix}-6\\8\end{pmatrix}$



a) Represent both of the above vectors diagramatically.

b) i) Calculate $|\mathbf{a}|$.

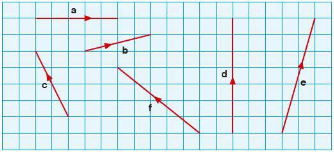
$$|\mathbf{a}| = \sqrt{(3^2 + 4^2)}$$
$$= \sqrt{25}$$
$$= 5$$

ii) Calculate $|\overrightarrow{BC}|$.

$$|\overrightarrow{BC}| = \sqrt{(-6)^2 + 8^2}$$
$$= \sqrt{100}$$
$$= 10$$

Exercise 5.4

1. Calculate the magnitude of the vectors shown below. Give your answers correct to 1 d.p.



2. Calculate the magnitude of the following vectors, giving your answers to 1 d.p.

a) $\overrightarrow{AB} = \begin{pmatrix}0\\4\end{pmatrix}$ b) $\overrightarrow{BC} = \begin{pmatrix}2\\5\end{pmatrix}$ c) $\overrightarrow{CD} = \begin{pmatrix}-4\\-6\end{pmatrix}$

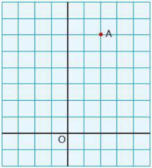
d) $\overrightarrow{DE} = \begin{pmatrix}-5\\12\end{pmatrix}$ e) $2\overrightarrow{AB}$ f) $-\overrightarrow{CD}$

3. $\mathbf{a} = \begin{pmatrix}4\\-3\end{pmatrix}$ $\mathbf{b} = \begin{pmatrix}-5\\7\end{pmatrix}$ $\mathbf{c} = \begin{pmatrix}-1\\-8\end{pmatrix}$

Calculate the magnitude of the following, giving your answers to 1 d.p.

a) $\mathbf{a} + \mathbf{b}$ b) $2\mathbf{a} - \mathbf{b}$ c) $\mathbf{b} - \mathbf{c}$

d) $2\mathbf{c} + 3\mathbf{b}$ e) $2\mathbf{b} - 3\mathbf{a}$ f) $\mathbf{a} + 2\mathbf{b} - \mathbf{c}$

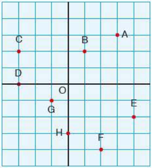


Position vectors

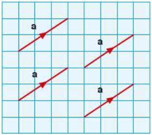
Sometimes a vector is fixed in position relative to a specific point. In the diagram (left), the position vector of A relative to O is $\begin{pmatrix}2\\6\end{pmatrix}$.

Exercise 5.5

1. Give the position vectors of A, B, C, D, E, F, G and H relative to O in the diagram shown.



Vector geometry

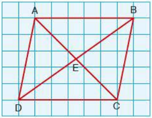


In general vectors are not fixed in position. If a vector **a** has a specific magnitude and direction, then any other vector with the same magnitude and direction as **a** can also be labelled **a**.

If $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then all the vectors shown in this diagram can also be labelled **a**, as they all have the same magnitude and direction.

This property of vectors can be used to solve problems in vector geometry.

Worked example



i) Name a vector equal to $\overrightarrow{AD}$.

$\overrightarrow{BC} = \overrightarrow{AD}$

ii) Write $\overrightarrow{BD}$ in terms of $\overrightarrow{BE}$.

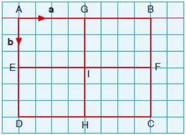
$\overrightarrow{BD} = 2\overrightarrow{BE}$

iii) Express $\overrightarrow{CD}$ in terms of $\overrightarrow{AB}$.

$\overrightarrow{CD} = \overrightarrow{BA} = -\overrightarrow{AB}$

Exercise 5.6

1.

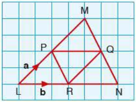


If $\overrightarrow{AG} = \mathbf{a}$ and $\overrightarrow{AE} = \mathbf{b}$, express the following in terms of **a** and **b**:

a) $\overrightarrow{EI}$ b) $\overrightarrow{HC}$ c) $\overrightarrow{FC}$

d) $\overrightarrow{DE}$ e) $\overrightarrow{GH}$ f) $\overrightarrow{CD}$

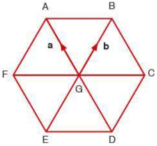
g) $\overrightarrow{AI}$ h) $\overrightarrow{GE}$ i) $\overrightarrow{FD}$



2. If $\overrightarrow{LP} = \mathbf{a}$ and $\overrightarrow{LR} = \mathbf{b}$, express the following in terms of **a** and **b**:

a) $\overrightarrow{LM}$ b) $\overrightarrow{PQ}$ c) $\overrightarrow{PR}$

d) $\overrightarrow{MQ}$ e) $\overrightarrow{MP}$ f) $\overrightarrow{NP}$



3. ABCDEF is a regular hexagon.

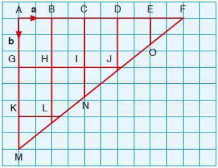
If $\overrightarrow{GA} = \mathbf{a}$ and $\overrightarrow{GB} = \mathbf{b}$, express the following in terms of **a** and **b**:

a) $\overrightarrow{AD}$ b) $\overrightarrow{FE}$ c) $\overrightarrow{DC}$

d) $\overrightarrow{AB}$ e) $\overrightarrow{FC}$ f) $\overrightarrow{EC}$

g) $\overrightarrow{BF}$ h) $\overrightarrow{FD}$ i) $\overrightarrow{AE}$

4.



If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AG} = \mathbf{b}$, express the following in terms of **a** and **b**:

a) $\overrightarrow{AF}$ b) $\overrightarrow{AM}$ c) $\overrightarrow{FM}$

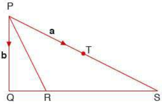
d) $\overrightarrow{FO}$ e) $\overrightarrow{EI}$ f) $\overrightarrow{KF}$

g) $\overrightarrow{CN}$ h) $\overrightarrow{AN}$ i) $\overrightarrow{DN}$

Exercise 5.7

1. In the diagram (below, right), T is the midpoint of the line PS and R divides the line QS in the ratio 1 : 3.

$\overrightarrow{PT} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.



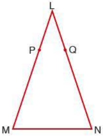
a) Express each of the following in terms of **a** and **b**:

i) $\overrightarrow{PS}$

ii) $\overrightarrow{QS}$

iii) $\overrightarrow{PR}$

b) Show that $\overrightarrow{RT} = \frac{1}{4}(2\mathbf{a} - 3\mathbf{b})$.

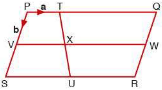


2. $\overrightarrow{PM} = 3\overrightarrow{LP}$ and $\overrightarrow{QN} = 3\overrightarrow{LQ}$

Prove that:

a) the line PQ is parallel to the line MN

b) the line MN is four times the length of the line PQ.



3. PQRS is a parallelogram (left). The point T divides the line PQ in the ratio 1 : 3, and U, V and W are the midpoints of SR, PS and QR respectively.

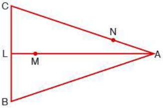
$\overrightarrow{PT} = \mathbf{a}$ and $\overrightarrow{PV} = \mathbf{b}$.

a) Express each of the following in terms of **a** and **b**:

i) $\overrightarrow{PQ}$ ii) $\overrightarrow{SU}$

iii) $\overrightarrow{PU}$ iv) $\overrightarrow{VX}$

b) Show that $\overrightarrow{XR} = \frac{1}{2}(5\mathbf{a} + 2\mathbf{b})$.



4. ABC is an isosceles triangle (left). L is the midpoint of BC. M divides the line LA in the ratio 1 : 5, and N divides AC in the ratio 2 : 5.

a) $\overrightarrow{BC} = \mathbf{p}$ and $\overrightarrow{BA} = \mathbf{q}$. Express the following in terms of **p** and **q**:

i) $\overrightarrow{LA}$ ii) $\overrightarrow{AN}$

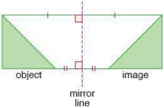
b) Show that $\overrightarrow{MN} = \frac{1}{84}(46\mathbf{q} - 11\mathbf{p})$.

SECTION 4 Transformations

An object undergoing a transformation changes either in position or shape. In its simplest form, this change can occur as a result of either a **reflection**, **rotation**, **translation** or **enlargement**. If an object undergoes a transformation, then its new position or shape is known as the **image**. The transformation that maps the image back onto the original object is known as an **inverse transformation**.

Reflection

If an object is reflected, it undergoes a 'flip' movement about a dashed (broken) line known as the **mirror line**, as shown in the diagram:

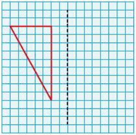


A point on the object and its equivalent point on the image are equidistant from the mirror line. This distance is measured at right angles to the mirror line. The line joining the point to its image is perpendicular to the mirror line.

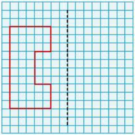
Exercise 5.8

Copy the following objects and mirror lines and, in each case, draw in the position of the object under reflection in the dashed mirror line(s).

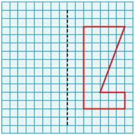
1.



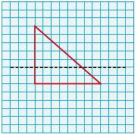
2.



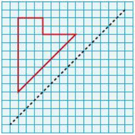
3.



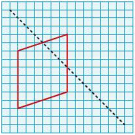
4.



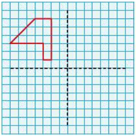
5.



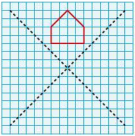
6.



7.



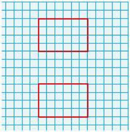
8.



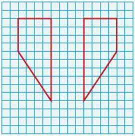
Exercise 5.9

Copy the following objects and images and, in each case, draw in the position of the mirror line(s).

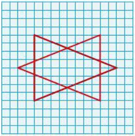
1.



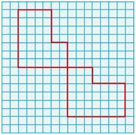
2.



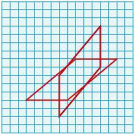
3.



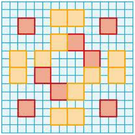
4.



5.



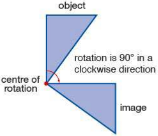
6.



Rotation

If an object is rotated it undergoes a 'turning' movement about a specific point known as the **centre of rotation**.

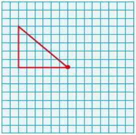
When describing a rotation, it is necessary to identify not only the position of the centre of rotation, but also the angle and direction of the turn, as shown in the diagram:



Exercise 5.10

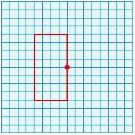
In the following, the object and centre of rotation have both been given. Copy each diagram and draw the object's image under the stated rotation about the marked point.

1.



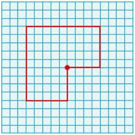
rotation 180°

2.



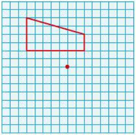
rotation 90° clockwise

3.



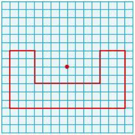
rotation 180°

4.



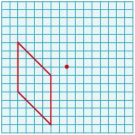
rotation 90° clockwise

5.



rotation 90° anti-clockwise

6.



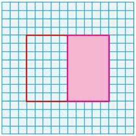
rotation 90° clockwise

Exercise 5.11

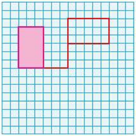
Copy the diagrams in Q. 1–6. In each case, the object (unshaded) and image (shaded) have been drawn.

a) Mark the centre of rotation.
b) Calculate the angle and direction of rotation.

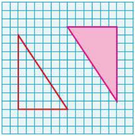
1.



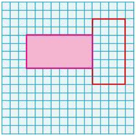
2.



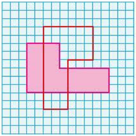
3.



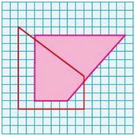
4.



5.



6.

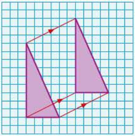


7. For each of the rotations above, give the inverse transformation.

Translation

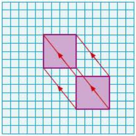
If an object is translated, it undergoes a 'straight sliding' movement. When describing a translation, it is necessary to give the translation vector. As no rotation is involved, each point on the object moves in the same way to its corresponding point on the image, e.g.

1.



Vector $= \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

2.

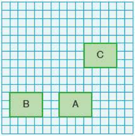


Vector $= \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

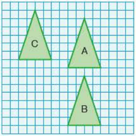
Exercise 5.12

In the following diagrams, object A has been translated to both of images B and C. Give the translation vectors.

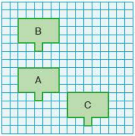
1.



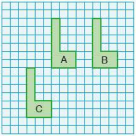
2.



3.



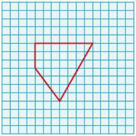
4.



Exercise 5.13

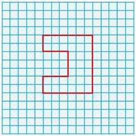
Copy each of the following diagrams. Translate the object by the vector given in each case and draw the image in its position. (Note that a bigger grid than the one shown may be needed.)

1.



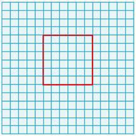
Vector $= \begin{pmatrix}3\\5\end{pmatrix}$

2.



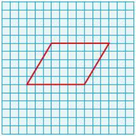
Vector $= \begin{pmatrix}5\\-4\end{pmatrix}$

3.



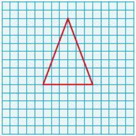
Vector $= \begin{pmatrix}-4\\6\end{pmatrix}$

4.



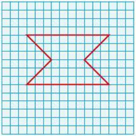
Vector = $\begin{pmatrix}-2\\-5\end{pmatrix}$

5.



Vector = $\begin{pmatrix}-6\\0\end{pmatrix}$

6.



Vector = $\begin{pmatrix}0\\-1\end{pmatrix}$

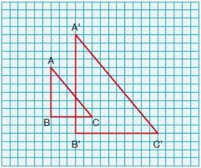
7. Give the vector that would map the image back on to the original object.

Enlargement

If an object is enlarged, the result is an image which is mathematically similar to the object but of a different size. The image can be either larger or smaller than the original object. When describing an enlargement two pieces of information need to be given, the position of the centre of enlargement and the scale factor of enlargement.

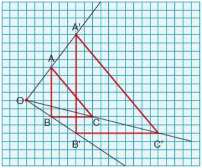
Worked examples

a) In the diagram below, triangle ABC is enlarged to form triangle A'B'C'.



i) Find the centre of enlargement.

The centre of enlargement is found by joining corresponding points on the object and image with a straight line. These lines are then extended until they meet. The point at which they meet is the centre of enlargement, O.



ii) Calculate the scale factor of enlargement.

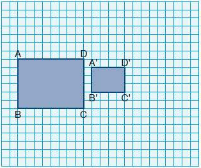
The scale factor of enlargement can be calculated in one of two ways. From the diagram above it can be seen that the distance OA' is twice the distance OA. Similarly OC' and OB' are both twice OC and OB respectively, hence the scale factor of enlargement is 2.

Alternatively the scale factor can be found by considering the ratio of the length of a side on the image to the length of the corresponding side on the object, i.e.

$$\frac{A'B'}{AB} = \frac{12}{6} = 2$$

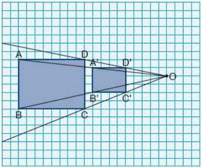
Hence the scale factor of enlargement is 2.

b) In the diagram below, the rectangle ABCD undergoes a transformation to form rectangle A'B'C'D'.



i) Find the centre of enlargement.

By joining corresponding points on both the object and the image, the centre of enlargement is found at O.



ii) Calculate the scale factor of enlargement.

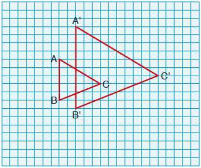
The scale factor of enlargement $= \frac{A'B'}{AB} = \frac{3}{6} = \frac{1}{2}$.

Note: If the scale factor of enlargement is greater than 1, then the image is larger than the object. If the scale factor lies between 0 and 1, then the resulting image is smaller than the object. In these cases, although the image is smaller than the object, the transformation is still known as an enlargement.

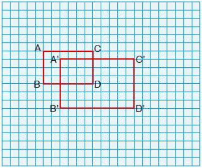
Exercise 5.14 Copy the following diagrams and find:

a) the centre of enlargement
b) the scale factor of enlargement.

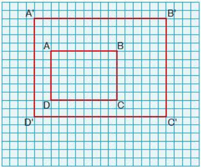
1.



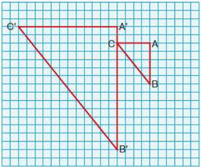
2.



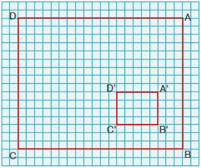
3.



4.



5.

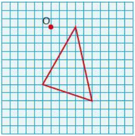


6. For each of Q.1–5, give the enlargement that would map the image back on to the original object.

Exercise 5.15

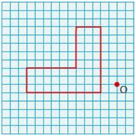
Copy each of the following diagrams. Enlarge the objects by the scale factor given and from the centre of enlargement shown. (Note that a bigger grid than the one shown may be needed.)

1.



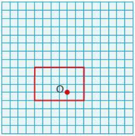
scale factor 2

2.



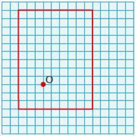
scale factor 2

3.



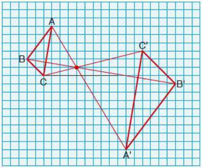
scale factor 3

4.



scale factor $\frac{1}{3}$

The diagram below shows an example of **negative enlargement**.

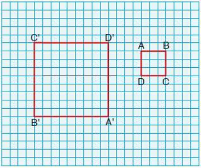


scale factor of enlargement is –2

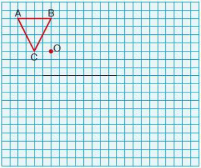
With negative enlargement, each point and its image are on opposite sides of the centre of enlargement. The scale factor of enlargement is calculated in the same way, remembering, however, to write a '–' sign before the number.

Exercise 5.16

1. Copy the following diagram, calculate the scale factor of enlargement and show the position of the centre of enlargement.

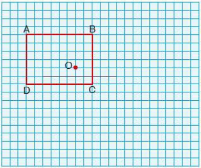


2. Copy the diagram and enlarge the object by the given scale factor and from the centre of enlargement shown.



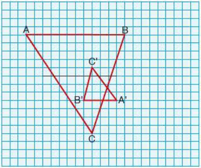
scale factor of enlargement is –2.5

3. Using the scale factor and centre of enlargement given, copy and complete the diagram.



scale factor of enlargement is –2

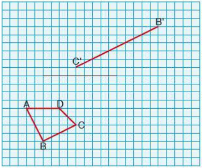
4. Copy the diagram. Find the scale factor of enlargement and mark the position of the centre of enlargement.



5. An object and part of its image under enlargement are given in the diagram below.

a) Copy the diagram and complete the image.

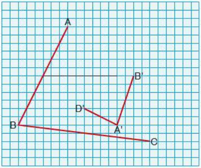
b) Mark the centre of enlargement and calculate the scale factor of enlargement.



6. In the diagram below, part of an object in the shape of a quadrilateral and its image under enlargement are drawn.

a) Copy and complete the diagram.

b) Mark the centre of enlargement and calculate the scale factor of enlargement.



SECTION 5

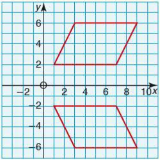
Further transformations

Section 4 introduced basic aspects of transformations. However, as with most branches of mathematics, a basic principle can be extended.

Reflection

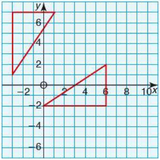
The position of the mirror line is essential when describing a reflection. At times its equation, as well as its position, will be required.

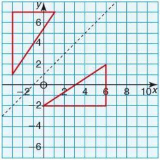
Worked examples



a) Find the equation of the mirror line in the reflection given in the diagram (left).
Here the mirror line is the *x*-axis. The equation of the mirror line is therefore $y = 0$.

b) A reflection is shown in the diagram below.





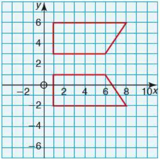
i) Draw the position of the mirror line.
ii) Give the equation of the mirror line.
$y = x + 1$.

Exercise 5.17

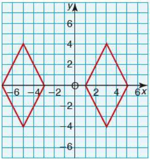
Copy each of the following diagrams, then:

a) draw the position of the mirror line(s)
b) give the equation of the mirror line(s).

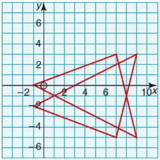
1.



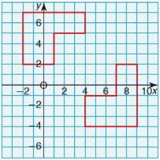
2.



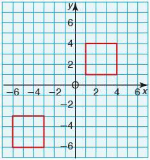
3.



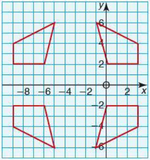
4.



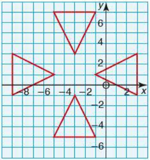
5.



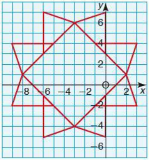
6.



7.



8.

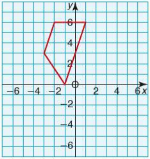


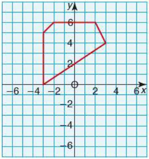
Exercise 5.18

In Q.1 and 2, copy both diagrams four times and reflect the object in each of the lines given.

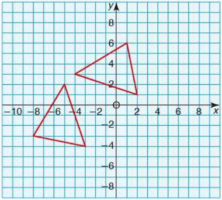
1. a) $x = 2$
b) $y = 0$
c) $y = x$
d) $y = -x$

2. a) $x = -1$
b) $y = -x - 1$
c) $y = x + 2$
d) $x = 0$

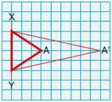




3. Copy the diagram (right), and reflect the triangles in the lines:
$x = 1$ and $y = -3$.



Stretch



A **stretch** is a transformation which distorts the shape of the object. There is a fixed line called the invariant line.

If an object undergoes a stretch, the effect is a lengthening in one direction only. When describing a stretch, two pieces of information are needed: the scale factor and the invariant line.

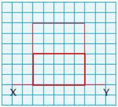
XY is the invariant line as the position of every point on it remains fixed. The perpendicular distance of A from the invariant line is 3 units. A' is the image of A after being stretched. The perpendicular distance of A' from the invariant line is 9 units.

$$\text{Scale factor} = \frac{\text{perpendicular distance of A' from XY}}{\text{perpendicular distance of A from XY}}$$
$$= \frac{9}{3}$$
$$= 3$$

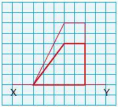
Exercise 5.19

In each the following, the object is outlined in **bold red**. XY is the invariant line. Calculate the scale factor for each of the stretches shown.

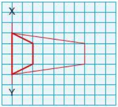
1.



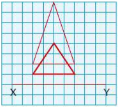
2.



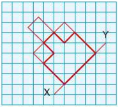
3.



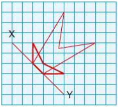
4.



5.

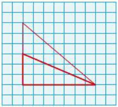


6.

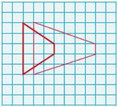


Exercise 5.20 In each of the following, both the object (in **bold red**) and the image have been drawn. Determine the position of the invariant line and calculate the stretch scale factor in each case.

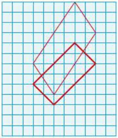
1.



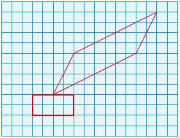
2.



3.



4.



Combinations of transformations

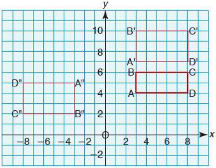
An object need not be subjected to just one type of transformation. It can undergo a succession of different transformations.

Worked examples

a) A rectangle ABCD maps onto A'B'C'D' after a stretch of scale factor 1.5, keeping the line $y = -2$ as invariant. A'B'C'D' maps onto A"B"C"D" after undergoing a rotation of 180° about the point (0, 6).

i) Draw and label the image A'B'C'D'.

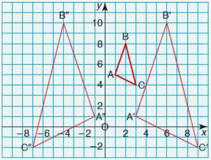
ii) Draw and label the image A"B"C"D".



b) A triangle ABC maps onto A'B'C' after an enlargement of scale factor 3 from the centre of enlargement (0, 7). A'B'C' is then mapped onto A"B"C" by a reflection in the line $x = 1$.

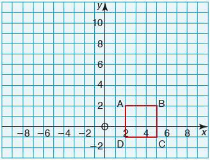
i) Draw and label the image A'B'C'.

ii) Draw and label the image A"B"C".

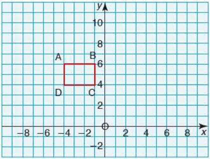


Exercise 5.21

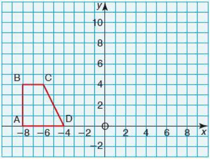
In each of the following questions, copy the diagram. After both transformations, draw the images on the same grid and label them clearly.



1. The square ABCD is mapped onto A'B'C'D' by a reflection in the line $y = 3$. A'B'C'D' then maps onto A"B"C"D" as a result of a 90° rotation in a clockwise direction about the point $(-2, 5)$.



2. The rectangle ABCD is mapped onto A'B'C'D' by an enlargement of scale factor -2 with its centre at $(0, 5)$. A'B'C'D' then maps onto A"B"C"D" as a result of a reflection in the line $y = -x + 7$.



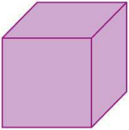
3. The trapezium ABCD is mapped onto A'B'C'D' by a stretch of scale factor 2 with $y = 0$ as the invariant line. A'B'C'D' then maps onto A"B"C"D" as a result of an enlargement of scale factor $-\frac{1}{2}$ with its centre at $(2, 4)$.

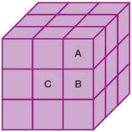
SECTION
6

Investigations, modelling and ICT

A painted cube

A $3 \times 3 \times 3$ cm cube is painted on the outside as shown in the left-hand diagram below:





The large cube is then cut up into 27 smaller cubes, each 1 cm × 1 cm × 1 cm as shown on the right.
$1 \times 1 \times 1$ cm cubes with 3 painted faces are labelled type A.
$1 \times 1 \times 1$ cm cubes with 2 painted faces are labelled type B.
$1 \times 1 \times 1$ cm cubes with 1 face painted are labelled type C.
$1 \times 1 \times 1$ cm cubes with no faces painted are labelled type D.

1. a) How many of the 27 cubes are type A?
b) How many of the 27 cubes are type B?
c) How many of the 27 cubes are type C?
d) How many of the 27 cubes are type D?

2. Consider a $4 \times 4 \times 4$ cm cube cut into $1 \times 1 \times 1$ cm cubes. How many of the cubes are type A, B, C and D?

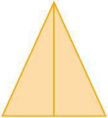
3. How many type A, B, C and D cubes are there when a $10 \times 10 \times 10$ cm cube is cut into $1 \times 1 \times 1$ cm cubes?

4. Generalise for the number of type A, B, C and D cubes in an $n \times n \times n$ cube.

5. Generalise for the number of type A, B, C and D cubes in a cuboid l cm long, w cm wide and h cm high.

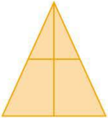
Triangle count

The diagram below shows an isosceles triangle with a vertical line drawn from its apex to its base.



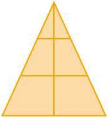
There is a total of 3 triangles in this diagram.

If a horizontal line is drawn across the triangle, it will look as shown:

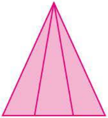


There is a total of 6 triangles in this diagram.

When one more horizontal line is added, the number of triangles increases further:



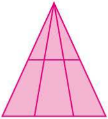
1. Calculate the total number of triangles in the diagram above with the two inner horizontal lines.
2. Investigate the relationship between the total number of triangles (t) and the number of inner horizontal lines (h). Enter your results in an ordered table.
3. Write an algebraic rule linking the total number of triangles and the number of inner horizontal lines.



The triangle (left) has two lines drawn from the apex to the base.

There is a total of 6 triangles in this diagram.

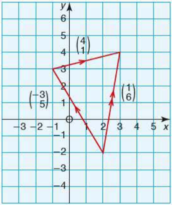
If a horizontal line is drawn through this triangle, the number of triangles increases as shown:



4. Calculate the total number of triangles in the diagram above with two lines from the vertex and one inner horizontal line.
5. Investigate the relationship between the total number of triangles (t) and the number of inner horizontal lines (h) when two lines are drawn from the apex. Enter your results in an ordered table.
6. Write an algebraic rule linking the total number of triangles and the number of inner horizontal lines.

ICT Activity

Using Autograph or another appropriate software package, prepare a help sheet for your revision that demonstrates the addition, subtraction and multiplication of vectors. An example is shown below:

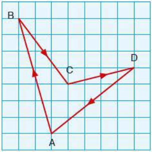


Vector addition:

$$\begin{pmatrix}-3\\5\end{pmatrix}+\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}$$

SECTION 7

Student assessments

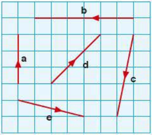


Student assessment 1

1. Using the diagram (left), describe the following translations with column vectors.

a) $\overrightarrow{AB}$ b) $\overrightarrow{DA}$ c) $\overrightarrow{CA}$

2. Describe each of the translations below using column vectors.



3. Using the vectors drawn in Q.2, draw diagrams to represent:

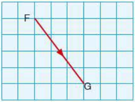
a) $\mathbf{a} + \mathbf{e}$ b) $\mathbf{c} - \mathbf{d}$
c) $-\mathbf{c} - \mathbf{e}$ d) $-\mathbf{b} + 2\mathbf{a}$

4. $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$.

Calculate:

a) $\mathbf{a} - \mathbf{c}$ b) $\mathbf{b} - \mathbf{a}$
c) $2\mathbf{a} + \mathbf{b}$ d) $3\mathbf{c} - 2\mathbf{a}$

Student assessment 2



1. a) Calculate the magnitude of the vector $\overrightarrow{FG}$ in the diagram.
b) Calculate the magnitude of each of the following vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 12 \\ -3 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}$$

2. $\mathbf{p} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

Calculate the magnitude of:

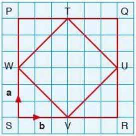
a) $4\mathbf{p} - \mathbf{r}$ b) $\frac{3}{2}\mathbf{q} - \mathbf{p}$

Give your answer to 1 d.p.

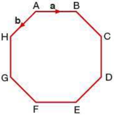
3. If $\overrightarrow{SW} = \mathbf{a}$ and $\overrightarrow{SV} = \mathbf{b}$ in the diagram (right), express each of the following in terms of **a** and **b**:

a) $\overrightarrow{SP}$ b) $\overrightarrow{QT}$

c) $\overrightarrow{TU}$



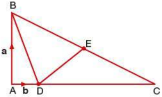
Student assessment 3



1. ABCDEFGH is a regular octagon.

$\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AH} = \mathbf{b}$. Express the following vectors in terms of **a** and **b**:

a) $\overrightarrow{FE}$ b) $\overrightarrow{ED}$ c) $\overrightarrow{BG}$

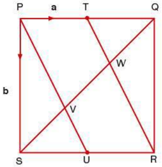


2. In the triangle ABC (left), $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. D divides the side AC in the ratio 1 : 4 and E is the midpoint of BC.

Express the following in terms of **a** and **b:**

a) $\overrightarrow{AC}$ b) $\overrightarrow{BC}$ c) $\overrightarrow{DE}$

3. In the square PQRS (below), T is the midpoint of the side PQ and U is the midpoint of the side SR. $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$.



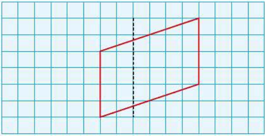
a) Express the following in terms of **a** and **b**:

i) $\overrightarrow{PT}$ ii) $\overrightarrow{QS}$

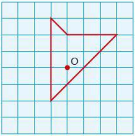
b) Calculate the ratio $\overrightarrow{PV} : \overrightarrow{PU}$.

Student assessment 4

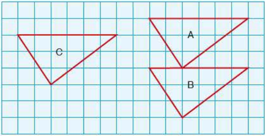
1. Reflect the object below in the mirror line shown.



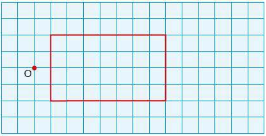
2. Rotate the object below 180° about the centre of rotation O.



3. Write down the column vector of the translation which maps:
a) triangle A to triangle B
b) triangle B to triangle C.



4. Enlarge the rectangle below by a scale factor 1.5 and from the centre of enlargement O.

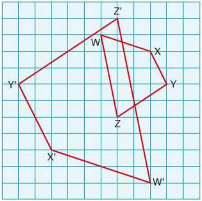


5. An object WXYZ and its image W'X'Y'Z' are shown below.

a) Find the position of the centre of enlargement.

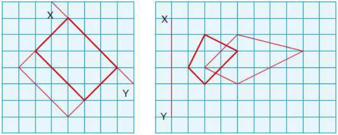
b) Calculate the scale factor of enlargement.

c) Determine the inverse transformation that maps the image back on to the original object.

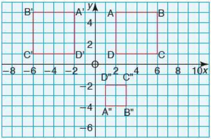


6. The objects below (in **bold red**) have been stretched. If XY is the invariant line, calculate the stretch factor for (a) and (b).

a) b)



7. Square ABCD is mapped onto square A'B'C'D'.
Square A'B'C'D' is then mapped onto square A"B"C"D".



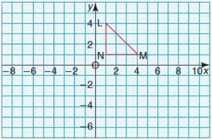
a) Describe fully the transformation which maps ABCD onto A'B'C'D'.

b) Describe fully the transformation which maps A'B'C'D' onto A"B"C"D".

8. Δ LMN below is mapped onto Δ L'M'N' by a stretch of scale factor 2 with $y = x + 3$ as the invariant line. Δ L'M'N' is then mapped onto Δ L"M"N" by a rotation of 180° about the point (0, 1).

a) Copy the diagram below, and plot and label the position of Δ L'M'N'.

b) On the same axes, plot and label the position of Δ L"M"N".



TOPIC 6

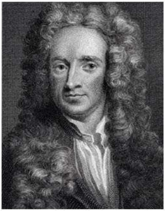
Mensuration

This topic will cover the following syllabus content:

6.3 Circumference and area of a circle
Arc length and area of sector
6.4 Surface area and volume of prism and pyramid (in particular, cuboid, cylinder and cone)
Surface area and volume of sphere
6.5 Areas and volumes of compound shapes

Sections

SECTION 1 The British



Isaac Newton (1642–1727)

Isaac Newton was born in Lincolnshire in 1642 and was probably the greatest scientist and mathematician ever to have lived. He was 22, and on leave from Cambridge University, when he began mathematical work on optics, dynamics, thermodynamics, acoustics and astronomy. He studied gravitation and the idea that white light is a mixture of all the rainbow's colours. He also designed the first reflecting telescope, the first reflecting microscope, and the sextant.

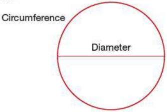
Newton is widely regarded as the 'Father of Calculus'. He discovered what is now called the Fundamental Theorem of Calculus, i.e. that integration and differentiation are each other's inverse operation. He applied calculus to solve many problems including finding areas, tangents, the lengths of curves and the maxima and minima of functions.

In 1687 Newton published *Philosophiae Naturalis Principia Mathematica*, one of the greatest scientific books ever written. The movement of the planets was not understood before Newton's Laws of Motion and the Law of Universal Gravitation. The idea that the Earth rotated about the Sun was introduced in ancient Greece, but Newton explained why this happens.

SECTION 2 Circumference and area of a circle

All circles are similar shapes. As a result, the ratio of their circumference to diameter is constant.

i.e.



$$\frac{\text{Circumference}}{\text{Diameter}} = \text{constant}$$

The constant is π(pi) which is 3.14 to 3 s.f.

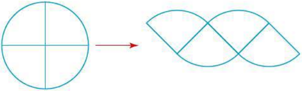
Therefore $\frac{C}{D} = \pi$

But the diameter is 2 × radius, so the above equation can be written as $\frac{C}{2r} = \pi$

So the circumference of a circle, $C = 2\pi r$

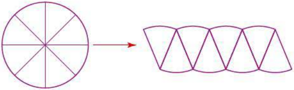
The area (A) of a circle can also be justified.

The diagram below shows a circle divided into four sectors. The sectors have then be rearranged and assembled as shown.

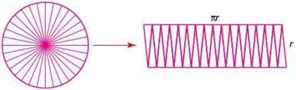


The total length of the curved edges is the same as the circumference of the circle.

If the circle is divided into eight sectors and each assembled as before, the diagram is:



As the number of sectors increases, the assembled shape begins to look more and more like a rectangle, as shown below with 32 sectors.

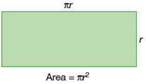


The top and bottom of the 'rectangle' is still equivalent to the circumference of the circle = $2\pi r$.

The top is therefore half the circumference = πr.

The height of the 'rectangle' is nearly equivalent to the radius of the circle.

With an infinite number of sectors, the circle can be rearranged to form a rectangle with a width πr and a height r.

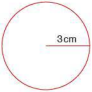


So the area of a circle, $A = \pi r^2$

Worked examples

NB: All diagrams are not drawn to scale.

a) Calculate the circumference of this circle, giving your answer to 3 s.f.



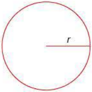
$C = 2\pi r$

$= 2\pi \times 3$

$= 18.8496$

The circumference is 18.8 cm.

b) If the circumference of this circle is 12 cm, calculate the radius, giving your answer to 3 s.f.



$C = 2\pi r$

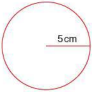
$r = \dfrac{C}{2\pi}$

$r = \dfrac{12}{2\pi}$

$= 1.90986$

The radius is 1.91 cm.

c) Calculate the area of this circle, giving your answer to 3 s.f.



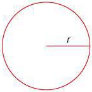
$A = \pi r^2$

$= \pi \times 5^2$

$= 78.5398$

The area is 78.5 cm^2.

d) If the area of this circle is 34 cm^2, calculate the radius, giving your answer to 3 s.f.



$A = \pi r^2$

$r = \sqrt{\dfrac{A}{\pi}}$

$r = \sqrt{\dfrac{34}{\pi}}$

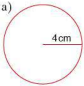
$= 3.2898$

The radius is 3.29 cm.

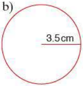
Exercise 6.1

1. Calculate the circumference of each circle, giving your answer to 3. s.f.

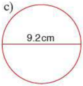
a)



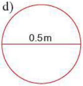
b)



c)



d)



2. Calculate the area of each of the circles in Q.1. Give your answers to 3 s.f.

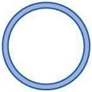
3. Calculate the radius of a circle when the circumference is:
   a) 15 cm b) π cm
   c) 4 m d) 8 mm

4. Calculate the diameter of a circle when the area is:
   a) 16 cm^2 b) 9π cm^2
   c) 8.2 m^2 d) 14.6 mm^2

Exercise 6.2

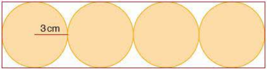
1. The wheel of a car has an outer radius of 25 cm. Calculate:
   i) how far the car has travelled after one complete turn of the wheel
   ii) how many times the wheel turns for a journey of 1 km.

2. If the wheel of a bicycle has a diameter of 60 cm, calculate how far a cyclist will have travelled after the wheel has rotated 100 times.



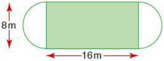
3. A circular ring has a cross-section as shown here. If the outer radius is 22 mm and the inner radius 20 mm, calculate the cross-sectional area of the ring.

4. Four circles are drawn in a line and enclosed by a rectangle as shown.



If the radius of each circle is 3 cm, calculate:
   a) the area of the rectangle
   b) the area of each circle
   c) the unshaded area within the rectangle.

5. A garden is made up of a rectangular patch of grass and two semicircular vegetable patches.

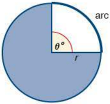


If the length and width of the rectangular patch are 16 m and 8 m respectively, calculate:
   a) the perimeter of the garden
   b) the total area of the garden.

SECTION 3

Arc length and area of a sector

NB: All diagrams are not drawn to scale.



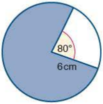
Arc length

An **arc** is part of the circumference of a circle between two radii. Its length is proportional to the size of the angle θ between the two radii. The length of the arc as a fraction of the circumference of the whole circle is therefore equal to the fraction that θ is of 360°.

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$

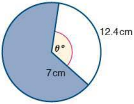
Worked examples

a) Find the length of the minor arc in the circle below. Give your answer to 1 d.p.



$$\text{Arc length} = \frac{80}{360} \times 2 \times \pi \times 6$$
$$= 8.4\,\text{cm}$$

b) In this circle, the length of the minor arc is 12.4 cm and the radius is 7 cm.



i) Calculate the angle $\theta°$.

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$
$$12.4 = \frac{\theta}{360} \times 2 \times \pi \times 7$$
$$\frac{12.4 \times 360}{2 \times \pi \times 7} = \theta$$
$$\theta = 101.5 \text{ (1 d.p)}$$

ii) Calculate the length of the major arc.

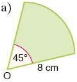
$$C = 2\pi r$$
$$= 2 \times \pi \times 7$$
$$= 44.0\,\text{cm (3 s.f.)}$$

$$\text{Major arc} = \text{circumference} - \text{minor arc}$$
$$= (44.0 - 12.4)\,\text{cm}$$
$$= 31.6\,\text{cm}$$

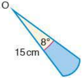
Exercise 6.3

1. For each of the following, give the length of the arc to 3 s.f. O is the centre of the circle.

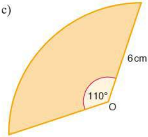
a)



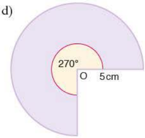
b)



c)



d)



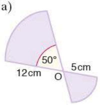
2. A sector is the region of a circle enclosed by two radii and an arc. Calculate the angle θ for each of the following sectors. The radius r and arc length a are given in each case.
   a) $r = 14$ cm, $a = 8$ cm
   b) $r = 4$ cm, $a = 16$ cm
   c) $r = 7.5$ cm, $a = 7.5$ cm
   d) $r = 6.8$ cm, $a = 13.6$ cm

3. Calculate the radius r for each of the following sectors. The angle θ and arc length a are given in each case.
   a) $\theta = 75°$, $a = 16$ cm
   b) $\theta = 300°$, $a = 24$ cm
   c) $\theta = 20°$, $a = 6.5$ cm
   d) $\theta = 243°$, $a = 17$ cm

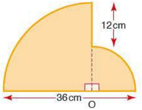
Exercise 6.4

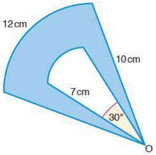
1. Calculate the perimeter of these shapes.

a)



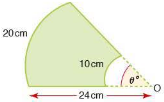
b)





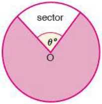
2. A shape is made from two sectors arranged in such a way that they share the same centre. The radius of the smaller sector is 7 cm and the radius of the larger sector is 10 cm.

   If the angle at the centre of the smaller sector is 30° and the arc length of the larger sector is 12 cm, calculate:
   a) the arc length of the smaller sector
   b) the total perimeter of the two sectors
   c) the angle at the centre of the larger sector.



3. For the diagram on the left, calculate:
   a) the radius of the smaller sector
   b) the perimeter of the shape
   c) the angle $\theta°$.

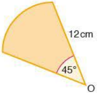
The area of a sector



A **sector** is the region of a circle enclosed by two radii and an arc. Its area is proportional to the size of the angle $\theta°$ between the two radii. The area of the sector as a fraction of the area of the whole circle is therefore equal to the fraction that $\theta°$ is of 360°.

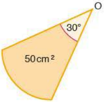
$$\text{Area of sector} = \frac{\theta}{360} \times \pi r^2$$

Worked examples



a) Calculate the area of the sector (right), giving your answer to 1 d.p.

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{45}{360} \times \pi \times 12^2$$
$$= 56.5\,\text{cm}^2$$



b) Calculate the radius of the sector (left), giving your answer to 3 s.f.

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$
$$50 = \frac{30}{360} \times \pi \times r^2$$
$$\frac{50 \times 360}{30\pi} = r^2$$
$$r = 13.8$$

The radius is 13.8 cm.

Exercise 6.5

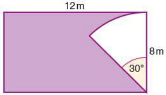
1. Calculate the area of each of the following sectors, using the values of the angles θ and radius r in each case.
   a) $\theta = 60°$, $r = 8\,\text{cm}$
   b) $\theta = 120°$, $r = 14\,\text{cm}$
   c) $\theta = 2°$, $r = 18\,\text{cm}$
   d) $\theta = 320°$, $r = 4\,\text{cm}$

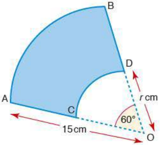
2. Calculate the radius for each of the following sectors, using the values of the angle θ and the area A in each case.
   a) $\theta = 40°$, $A = 120\,\text{cm}^2$
   b) $\theta = 12°$, $A = 42\,\text{cm}^2$
   c) $\theta = 150°$, $A = 4\,\text{cm}^2$
   d) $\theta = 300°$, $A = 400\,\text{cm}^2$

3. Calculate the value of the angle θ, to the nearest degree, or each of the following sectors, using the values of A and r in each case.
   a) $r = 12\,\text{cm}$, $A = 60\,\text{cm}^2$
   b) $r = 26\,\text{cm}$, $A = 0.02\,\text{m}^2$
   c) $r = 0.32\,\text{m}$, $A = 180\,\text{cm}^2$
   d) $r = 38\,\text{mm}$, $A = 16\,\text{cm}^2$

Exercise 6.6

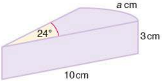
1. A rotating sprinkler is placed in one corner of a garden as shown. If it has a reach of 8 m and rotates through an angle of 30°, calculate the area of garden not being watered.

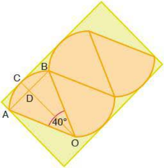




2. Two sectors AOB and COD share the same centre O. The area of AOB is three times the area of COD. Calculate:
   a) the area of sector AOB
   b) the area of sector COD
   c) the radius r cm of sector COD.

3. A circular cake is cut. One of the slices is shown. Calculate:
   a) the length a cm of the arc
   b) the total surface area of all the sides of the slice.



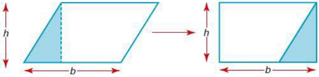


4. The diagram (left) shows a plan view of four tiles in the shape of sectors placed in the bottom of a box. C is the midpoint of the arc AB and intersects the chord AB at point D. If the length ADB is 8 cm and the length OB is 10 cm, calculate:
   a) the length OD
   b) the length CD
   c) the area of the sector AOB
   d) the length and width of the box
   e) the area of the base of the box not covered by the tiles.

SECTION 4 Area and volume of plane shapes and prisms

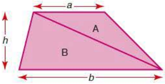
The area of parallelograms and trapeziums

A **parallelogram** can be rearranged to form a rectangle in the way shown below:



Therefore, area of parallelogram = base length × perpendicular height.

A **trapezium** can be visualised as being split into two triangles as shown (left):



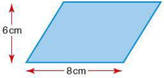
Area of triangle A $= \frac{1}{2} \times a \times h$

Area of triangle B $= \frac{1}{2} \times b \times h$

Area of the trapezium = area of triangle A + area of triangle B

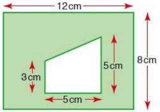
$$= \tfrac{1}{2}ah + \tfrac{1}{2}bh$$
$$= \tfrac{1}{2}h(a + b)$$

Worked examples **a)** Calculate the area of the parallelogram shown below:



Area = base length × perpendicular height
$= 8 \times 6$
$= 48\,\text{cm}^2$

b) Calculate the shaded area in the shape shown (left):



Area of rectangle $= 12 \times 8$
$= 96\,\text{cm}^2$

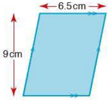
Area of trapezium $= \frac{1}{2} \times 5(3 + 5)$
$= 2.5 \times 8$
$= 20\,\text{cm}^2$

Shaded area $= 96 - 20$
$= 76\,\text{cm}^2$

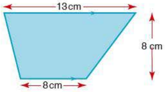
Exercise 6.7

Find the area of each of the following shapes:

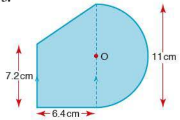
1.



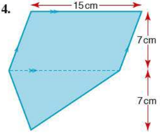
2.



3.

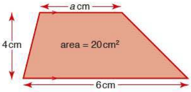


4.

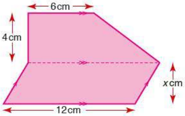


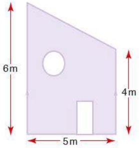
Exercise 6.8

1. Calculate *a*.



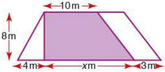
2. If the areas of this trapezium and parallelogram are equal, calculate *x*.





3. The end view of a house is as shown in the diagram (left).

   If the door has a width and height of 0.75 m and 2 m respectively and the circular window has a diameter of 0.8 m, calculate the area of brickwork.

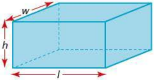


4. A garden in the shape of a trapezium is split into three parts: two flower beds in the shape of a triangle and a parallelogram and a section of grass in the shape of a trapezium.

The area of the grass is two and a half times the total area of flower beds. Calculate:

a) the area of each flower bed
b) the area of grass
c) the value of *x*.

The surface area of a cuboid and cylinder

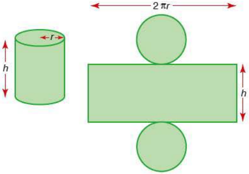


To calculate the surface area of a **cuboid**, start by looking at its individual faces. These are either squares or rectangles. The surface area of a cuboid is the sum of the areas of its faces.

Area of top $= wl$	Area of bottom $= wl$
Area of front $= lh$	Area of back $= lh$
Area of one side $= wh$	Area of other side $= wh$

Total surface area
$= 2wl + 2lh + 2wh$
$= 2(wl + lh + wh)$

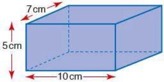
For the surface area of a **cylinder**, it is best to visualise the net of the solid: it is made up of one rectangular piece and two circular pieces.



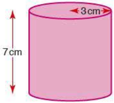
Area of circular pieces $= 2 \times \pi r^2$
Area of rectangular piece $= 2\pi r \times h$
Total surface area $= 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$

Worked examples

a) Calculate the surface area of the cuboid shown below:



Total area of top and bottom $= 2 \times 7 \times 10 = 140\,\text{cm}^2$
Total area of front and back $= 2 \times 5 \times 10 = 100\,\text{cm}^2$
Total area of both sides $= 2 \times 5 \times 7 = 70\,\text{cm}^2$
Total surface area $= 310\,\text{cm}^2$



b) If the height of a cylinder is 7cm and the radius of its circular top is 3cm, calculate its surface area.

$$\begin{aligned}\text{Total surface area} &= 2\pi r(r+h)\\ &= 2\pi \times 3 \times (3+7)\\ &= 6\pi \times 10\\ &= 60\pi\\ &= 188\,\text{cm}^2 \text{ (3 s.f.)}\end{aligned}$$

The total surface area is $188\,\text{cm}^2$.

Exercise 6.9

1. Calculate the surface area of each of the following cuboids if:

a) $l = 12\,\text{cm}$, $w = 10\,\text{cm}$, $h = 5\,\text{cm}$
b) $l = 4\,\text{cm}$, $w = 6\,\text{cm}$, $h = 8\,\text{cm}$
c) $l = 4.2\,\text{cm}$, $w = 7.1\,\text{cm}$, $h = 3.9\,\text{cm}$
d) $l = 5.2\,\text{cm}$, $w = 2.1\,\text{cm}$, $h = 0.8\,\text{cm}$

2. Calculate the height of each of the following cuboids if:

a) $l = 5\,\text{cm}$, $w = 6\,\text{cm}$, surface area $= 104\,\text{cm}^2$
b) $l = 2\,\text{cm}$, $w = 8\,\text{cm}$, surface area $= 112\,\text{cm}^2$
c) $l = 3.5\,\text{cm}$, $w = 4\,\text{cm}$, surface area $= 118\,\text{cm}^2$
d) $l = 4.2\,\text{cm}$, $w = 10\,\text{cm}$, surface area $= 226\,\text{cm}^2$

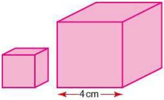
3. Calculate the surface area of each of the following cylinders if:

a) $r = 2\,\text{cm}, h = 6\,\text{cm}$
b) $r = 4\,\text{cm}, h = 7\,\text{cm}$
c) $r = 3.5\,\text{cm}, h = 9.2\,\text{cm}$
d) $r = 0.8\,\text{cm}, h = 4.3\,\text{cm}$

4. Calculate the height of each of the following cylinders. Give your answers to 1 d.p.

a) $r = 2.0\,\text{cm}$, surface area $= 40\,\text{cm}^2$
b) $r = 3.5\,\text{cm}$, surface area $= 88\,\text{cm}^2$
c) $r = 5.5\,\text{cm}$, surface area $= 250\,\text{cm}^2$
d) $r = 3.0\,\text{cm}$, surface area $= 189\,\text{cm}^2$

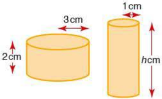
Exercise 6.10



1. Two cubes (left) are placed next to each other.

   The length of each of the edges of the larger cube is 4 cm. If the ratio of their surface areas is 1 : 4, calculate:
   a) the surface area of the small cube
   b) the length of an edge of the small cube.

2. A cube and a cylinder have the same surface area. If the cube has an edge length of 6 cm and the cylinder a radius of 2 cm, calculate:
   a) the surface area of the cube
   b) the height of the cylinder.



3. The two cylinders (left) have the same surface area. The shorter of the two has a radius of 3 cm and a height of 2 cm, and the taller cylinder has a radius of 1 cm.

   Calculate:
   a) the surface area of one of the cylinders
   b) the height h of the taller cylinder.

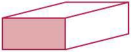
4. Two cuboids have the same surface area. The dimensions of one of them are: length = 3 cm, width = 4 cm and height = 2 cm.

   Calculate the height of the other cuboid if its length is 1 cm and width is 4 cm.

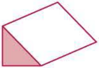
The volume of prisms

A prism is any three-dimensional object which has a constant cross-sectional area.

Below are a few examples of some of the more common types of prisms:


Rectangular prism
(cuboid)


Circular prism
(cylinder)

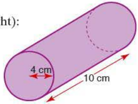

Triangular prism

When each of the shapes is cut parallel to the shaded face, the cross-section is constant and the shape is therefore classified as a prism.

Volume of a prism = area of cross-section × length

Worked examples

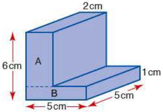
a) Calculate the volume of the cylinder shown in the diagram (right):



Volume = cross-sectional area $\times$ length

$= \pi \times 4^2 \times 10$

Volume = 503 cm^3 (3 s.f.)



b) Calculate the volume of the 'L' shaped prism shown in the diagram (left):

The cross-sectional area can be split into two rectangles:

Area of rectangle A = 5×2

$= 10\,cm^2$

Area of rectangle B = 5×1

$= 5\,cm^2$

Total cross-sectional area = $(10\,cm^2 + 5\,cm^2) = 15\,cm^2$

Volume of prism = 15×5

$= 75\,cm^3$

Exercise 6.11

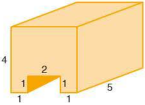
1. Calculate the volume of each of the following cuboids:
- a) Width 2 cm, Length 3 cm, Height 4 cm
- b) Width 6 cm, Length 1 cm, Height 3 cm
- c) Width 6 cm, Length 23 mm, Height 2 cm
- d) Width 42 mm, Length 3 cm, Height 0.007 m

2. Calculate the volume of each of the following cylinders:
- a) Radius 4 cm, Height 9 cm
- b) Radius 3.5 cm, Height 7.2 cm
- c) Radius 25 mm, Height 10 cm
- d) Radius 0.3 cm, Height 17 mm

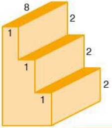
3. Calculate the volume of each of the following triangular prisms:
- a) Base length 6 cm
  Perpendicular height 3 cm
  Length 12 cm
- b) Base length 4 cm
  Perpendicular height 7 cm
  Length 10 cm
- c) Base length 5 cm
  Perpendicular height 24 mm
  Length 7 cm
- d) Base length 62 mm
  Perpendicular height 2 cm
  Length 0.01 m

4. Calculate the volume of each of the following prisms. All dimensions are given in centimetres.

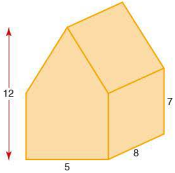
a)



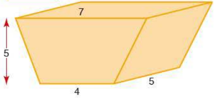
b)



c)

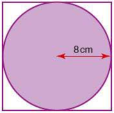


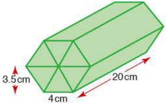
d)



Exercise 6.12

1. The diagram shows a plan view of a cylinder inside a box the shape of a cube. If the radius of the cylinder is 8 cm, calculate:
   a) the height of the cube
   b) the volume of the cube
   c) the volume of the cylinder
   d) the percentage volume of the cube not occupied by the cylinder.



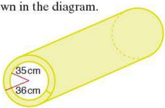


2. A chocolate bar is made in the shape of a triangular prism. The triangular face of the prism is equilateral and has an edge length of 4 cm and a perpendicular height of 3.5 cm. The manufacturer also sells these in special packs of six bars arranged as a hexagonal prism.

   If the prisms are 20 cm long, calculate:
   a) the cross-sectional area of the pack
   b) the volume of the pack.

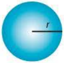
3. A cuboid and a cylinder have the same volume. The radius and height of the cylinder are 2.5 cm and 8 cm respectively. If the length and width of the cuboid are each 5 cm, calculate its height to 1 d.p.

4. A section of steel pipe is shown in the diagram. The inner radius is 35 cm and the outer radius 36 cm. Calculate the volume of steel used in making the pipe if it has a length of 130 m.



SECTION 5

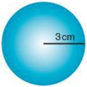
Surface area and volume of other solids



Volume of a sphere

Volume of sphere = $\frac{4}{3}\pi r^3$

Worked examples



a) Calculate the volume of the sphere on the left, giving your answer to 1 d.p.

$$\begin{aligned}\text{Volume of sphere} &= \tfrac{4}{3}\pi r^3\\ &= \tfrac{4}{3}\times\pi\times 3^3\\ &= 113.1\end{aligned}$$

The volume is 113.1 cm^3.

b) Given that the volume of a sphere is 150 cm^3, calculate its radius to 1 d.p.

$$V = \tfrac{4}{3}\pi r^3$$

$$r^3 = \frac{3V}{4\pi}$$

$$r^3 = \frac{3\times 150}{4\times\pi}$$

$$r = \sqrt[3]{35.8} = 3.3$$

The radius is 3.3 cm.

Exercise 6.13

1. Calculate the volume of each of the following spheres. The radius is given in each case.

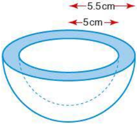
a) 6 cm b) 9.5 cm
c) 8.2 cm d) 0.7 cm

2. Calculate the radius of each of the following spheres. Give your answers in centimetres and to 1 d.p. The volume is given in each case.

a) 130 cm^3 b) 720 cm^3
c) 0.2 m^3 d) 1000 mm^3

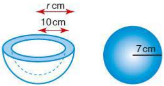
Exercise 6.14

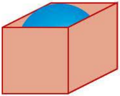
1. Given that sphere B has twice the volume of sphere A, calculate the radius of sphere B. Give your answer to 1 d.p.



2. Calculate the volume of material used to make the hemispherical bowl shown (left), if the inner radius of the bowl is 5 cm and its outer radius 5.5 cm.

3. The volume of the material used to make the sphere and hemispherical bowl below are the same. Given that the radius of the sphere is 7 cm and the inner radius of the bowl is 10 cm, calculate, to 1 d.p., the outer radius *r* cm of the bowl.

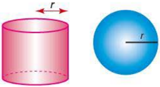




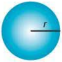
4. A ball is placed inside a box into which it will fit tightly. If the radius of the ball is 10 cm, calculate:
   a) the volume of the ball
   b) the volume of the box
   c) the percentage volume of the box not occupied by the ball.

5. A steel ball is melted down to make eight smaller identical balls. If the radius of the original steel ball was 20 cm, calculate to the nearest millimetre the radius of each of the smaller balls.

6. A steel ball of volume 600 cm^3 is melted down and made into three smaller balls, A, B and C. If the volumes of A, B and C are in the ratio 7 : 5 : 3, calculate to 1 d.p. the radius of each of A, B and C.



7. The cylinder and sphere shown (left) have the same radius and the same height. Calculate the ratio of their volumes, giving your answer in the form, volume of cylinder : volume of sphere.



The surface area of a sphere

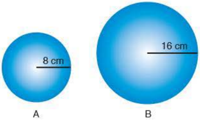
Surface area of sphere = $4\pi r^2$

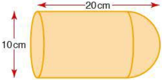
Exercise 6.15

1. Calculate the surface area of each of the following spheres. The radius is given in each case.
   a) 6 cm b) 4.5 cm
   c) 12.25 cm d) $\frac{1}{\sqrt{\pi}}$ cm

2. Calculate the radius of each of the following spheres. The surface area is given in each case.
   a) $50\,\text{cm}^2$ b) $16.5\,\text{cm}^2$
   c) $120\,\text{mm}^2$ d) $\pi\,\text{cm}^2$

3. Sphere A has a radius of 8 cm and sphere B has a radius of 16 cm. Calculate the ratio of their surface areas in the form 1 : *n*.

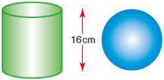




4. A hemisphere of diameter 10 cm is attached to a cylinder of equal diameter as shown (left).

   If the total length of the shape is 20 cm, calculate:
   a) the surface area of the hemisphere
   b) the length of the cylinder
   c) the surface area of the whole shape.

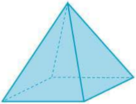
5. A sphere and a cylinder both have the same surface area and the same height of 16 cm.



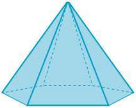
   Calculate:
   a) the surface area of the sphere,
   b) the radius of the cylinder.

The volume of a pyramid

A pyramid is a three-dimensional shape in which each of its faces must be plane. A pyramid has a polygon for its base and the other faces are triangles with a common vertex, known as the **apex**. Its individual name is taken from the shape of the base.



Square-based pyramid

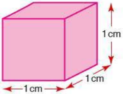


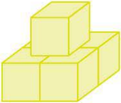
Hexagonal-based pyramid

To derive the formula for the volume of a pyramid requires mathematics at a higher level than covered in this book. However, below are diagrams to show how an approximate value for the volume of a pyramid is derived.

Consider first a cube of side length 1 cm.

Its volume is 1 cm^3.



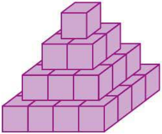


Now consider a step pyramid of two layers made of cubes of side length $\frac{1}{2}$ cm.

Volume of top layer $\left(\frac{1}{2}\right)^3 = \frac{1}{8}\text{cm}^3$

Volume of second layer $\frac{1}{2} \times 1 \times 1 = \frac{1}{2}\text{cm}^3$

Total volume $= \frac{1}{2} + \frac{1}{8} = \frac{5}{8}\text{cm}^3 = 0.625\,\text{cm}^3$



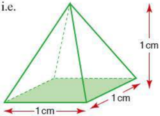
Now consider a step pyramid of four layers, made of cubes of side length $\frac{1}{4}$ cm.

Volume of top layer $\left(\frac{1}{4}\right)^3 = \frac{1}{64}\text{cm}^3$
Volume of second layer $= \frac{1}{4} \times \frac{2}{4} \times \frac{2}{4} = \frac{4}{64}\text{cm}^3$
Volume of third layer $= \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{64}\text{cm}^3$
Volume of bottom layer $= \frac{1}{4} \times \frac{4}{4} \times \frac{4}{4} = \frac{16}{64}\text{cm}^3$
So the total volume $= \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} + \frac{30}{64}\text{cm}^3 \approx 0.469\ \text{cm}^3$

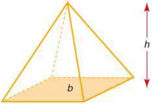
A step pyramid of 10 layers, made of cubes of side length $\frac{1}{10}$ cm can be shown to have a total volume of $\frac{77}{200}\text{cm}^3 = 0.385\,\text{cm}^3$

In fact as the number of layers increases the total volume for a step cube of total height 1 unit and base length and width of 1 unit gets closer and closer to $\frac{1}{3}$ units^3.

i.e.



Volume = $\frac{1}{3} \times 1 \times 1 \times 1 = \frac{1}{3}$cm^3

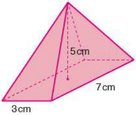


In general, for any pyramid:

Volume = $\frac{1}{3}$ × area of base × perpendicular height

Volume = $\frac{1}{3}bh$

Worked examples

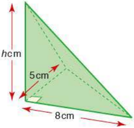


a) A rectangular-based pyramid has a perpendicular height of 5 cm and base dimensions as shown. Calculate the volume of the pyramid.

$$\begin{aligned} \text{Volume} &= \tfrac{1}{3} \times \text{base area} \times \text{height} \\ &= \tfrac{1}{3} \times 3 \times 7 \times 5 \\ &= 35 \end{aligned}$$

The volume is 35 cm^3.

b) The pyramid shown below has a volume of 60 cm^3. Calculate its perpendicular height h cm.



$$\text{Volume} = \tfrac{1}{3} \times \text{base area} \times \text{height}$$

$$\text{Height} = \frac{3 \times \text{volume}}{\text{base area}}$$

$$h = \frac{3 \times 60}{\frac{1}{2} \times 8 \times 5}$$

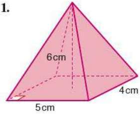
$$h = 9$$

The height is 9 cm.

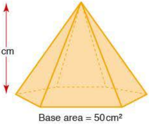
Exercise 6.16

Find the volume of each of the following pyramids:

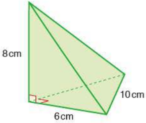
1.



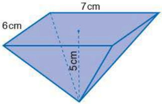
2.



3.

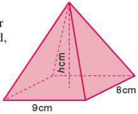


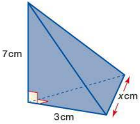
4.



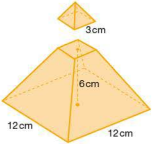
Exercise 6.17

1. Calculate the perpendicular height h cm for the pyramid, given that it has a volume of 168 cm^3.





2. Calculate the length of the edge marked x cm, given that the volume of the pyramid (left) is 14 cm^3.
3. The top of a square-based pyramid (below) is cut off.



The cut is made parallel to the base. If the base of the smaller pyramid has a side length of 3 cm and the vertical height of the truncated pyramid is 6 cm, calculate:

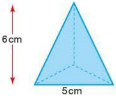
a) the height of the original pyramid
b) the volume of the original pyramid
c) the volume of the truncated pyramid.

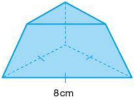
4. The top of a triangular-based pyramid (tetrahedron) is cut off. The cut is made parallel to the base. If the vertical height of the top is 6 cm, calculate:

a) the height of the truncated piece

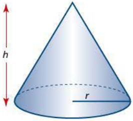
b) the volume of the small pyramid

c) the volume of the original pyramid.





The volume of a cone

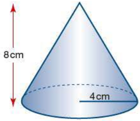


A cone is a pyramid with a circular base. The formula for its volume is therefore the same as for any other pyramid.

$$\text{Volume} = \tfrac{1}{3} \times \text{base area} \times \text{height}$$
$$= \tfrac{1}{3} \times \pi r^2 h$$

Worked examples

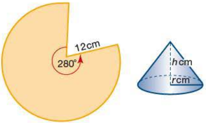
a) Calculate the volume of the cone (left).



$$\text{Volume} = \tfrac{1}{3}\pi r^2 h$$
$$= \tfrac{1}{3} \times \pi \times 4^2 \times 8$$
$$= 134 \text{ (3 s.f.)}$$

The volume is 134 cm$^{3.}$

b) The sector below is assembled to form a cone as shown:



i) Calculate the base circumference of the cone.
The base circumference of the cone is equal to the arc length of the sector.

$$\text{Sector arc length} = \frac{\theta}{360} \times 2\pi r$$
$$= \frac{280}{360} \times 2\pi \times 12 = 58.6 \text{ (3 s.f.)}$$

So the base circumference is 58.6 cm.

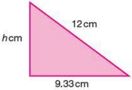
ii) Calculate the base radius of the cone.

The base of a cone is circular, therefore:

$$C = 2\pi r$$
$$r = \frac{C}{2\pi} = \frac{58.6}{2\pi}$$
$$= 9.33 \text{ (3 s.f.)}$$

So the radius is 9.33 cm.

iii) Calculate the vertical height of the cone.



The vertical height of the cone can be calculated using Pythagoras' theorem on the right-angled triangle enclosed by the base radius, vertical height and the sloping face shown:

Note that the length of the sloping face is equal to the radius of the sector.

$$12^2 = h^2 + 9.33^2$$
$$h^2 = 12^2 - 9.33^2$$
$$h^2 = 56.9$$
$$h = 7.54 \text{ (3 s.f.)}$$

So the height is 7.54 cm.

iv) Calculate the volume of the cone.

$$\text{Volume} = \tfrac{1}{3} \times \pi r^2 h$$
$$= \tfrac{1}{3} \times \pi \times 9.33^2 \times 7.54$$
$$= 688 \text{ (3 s.f.)}$$

So the volume is 688 cm^3.

It is important to note that, although answers were given to 3 s.f. in each case, where the answer was needed in a subsequent calculation the exact value was used and not the rounded one. By doing this we avoid introducing rounding errors into the calculations.

Exercise 6.18

1. Calculate the volume of each of the following cones. Use the values for the base radius *r* and the vertical height *h* given in each case.

a) $r = 3$ cm, $h = 6$ cm
b) $r = 6$ cm, $h = 7$ cm
c) $r = 8$ mm, $h = 2$ cm
d) $r = 6$ cm, $h = 44$ mm

2. Calculate the base radius of each of the following cones. Use the values for the volume V and the vertical height h given in each case.

a) $V = 600\,\text{cm}^3$, $h = 12\,\text{cm}$
b) $V = 225\,\text{cm}^3$, $h = 18\,\text{mm}$
c) $V = 1400\,\text{mm}^3$, $h = 2\,\text{cm}$
d) $V = 0.04\,\text{m}^3$ $h = 145\,\text{mm}$

3. The base circumference C and the length of the sloping face l is given for each of the following cones. Calculate:

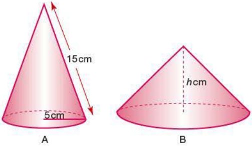
i) the base radius
ii) the vertical height
iii) the volume in each case.

Give all answers to 3 s.f.

a) $C = 50\,\text{cm}$, $l = 15\,\text{cm}$
b) $C = 100\,\text{cm}$, $l = 18\,\text{cm}$
c) $C = 0.4\,\text{m}$, $l = 75\,\text{mm}$
d) $C = 240\,\text{mm}$, $l = 6\,\text{cm}$

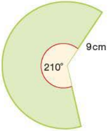
Exercise 6.19

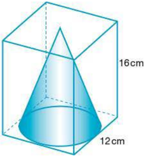
1. The two cones A and B shown below have the same volume. Using the dimensions shown and given that the base circumference of cone B is 60 cm, calculate the height h cm.



2. The sector shown (left) is assembled to form a cone. Calculate:

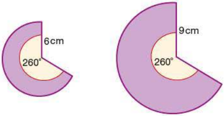
a) the base circumference of the cone
b) the base radius of the cone
c) the vertical height of the cone
d) the volume of the cone
e) the curved surface area of the cone.





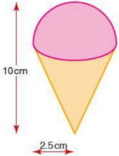
3. A cone is placed inside a cuboid as shown.
   If the base diameter of the cone is 12 cm and the height of the cuboid is 16 cm, calculate:
   a) the volume of the cuboid
   b) the volume of the cone
   c) the volume of the cuboid not occupied by the cone.

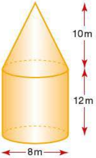
4. Two similar sectors are assembled into cones (below).
   Calculate:
   a) the volume of the smaller cone
   b) the volume of the larger cone
   c) the ratio of their volumes.



Exercise 6.20

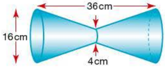
1. An ice cream consists of a hemisphere and a cone. Calculate its total volume.





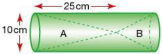
2. A cone is placed on top of a cylinder (left). Using the dimensions given, calculate the total volume of the shape.

3. Two identical truncated cones are placed end to end as shown:



Calculate the total volume of the shape.

4. Two cones A and B are placed either end of a cylindrical tube as shown.



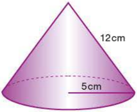
Given that the volumes of A and B are in the ratio 2 : 1, calculate:

a) the volume of cone A
b) the height of cone B
c) the volume of the cylinder.

The surface area of a cone

The surface area of a cone comprises the area of the circular base and the area of the curved face. The area of the curved face is equal to the area of the sector from which it is formed.

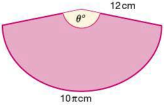
Worked example Calculate the total surface area of the cone shown (left):



Surface area of base $= \pi r^2$
$= 25\pi \text{ cm}^2$

The curved surface area can best be visualised if drawn as a sector as shown in the diagram (bottom left):

The radius of the sector is equivalent to the slant height of the cone. The curved perimeter of the sector is equivalent to the base circumference of the cone.



$$\frac{\theta}{360} = \frac{10\pi}{24\pi}$$

Therefore $\theta = 150$

$$\text{Area of sector} = \frac{150}{360} \times \pi \times 12^2 = 60\pi \text{ cm}$$

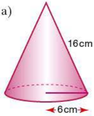
Total surface area $= 60\pi + 25\pi$
$= 85\pi$
$= 267(3 \text{ s.f.})$

The total surface area is 267 cm^2.

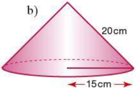
Exercise 6.21

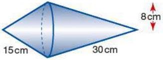
1. Calculate the surface area of the following cones:

a)



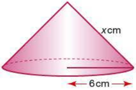
b)





2. Two cones with the same base radius are stuck together as shown on the left.
Calculate the surface area of the shape.

3. Two cones have the same total surface area (below).



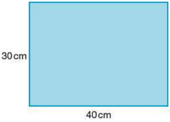
Calculate:
a) the total surface area of each cone
b) the value of x.

SECTION 6

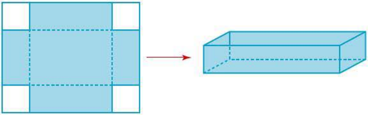
Investigations, modelling and ICT

Metal trays

A rectangular sheet of metal measures 30 × 40cm.



The sheet has squares of equal size cut from each corner. It is then folded to form a metal tray as shown.

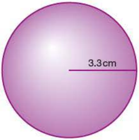


1. a) Calculate the length, width and height of the tray if a square of side length 1 cm is cut from each corner of the sheet of metal.
   b) Calculate the volume of this tray.
2. a) Calculate the length, width and height of the tray if a square of side length 2 cm is cut from each corner of the sheet of metal.
   b) Calculate the volume of this tray.
3. Using a spreadsheet if necessary, investigate the relationship between the volume of the tray and the size of the square cut from each corner. Enter your results in an ordered table.
4. Calculate, to 1 d.p. the side length of the square that produces the tray with the greatest volume.
5. State the greatest volume to the nearest whole number.

Tennis balls

Tennis balls are spherical and have a radius of 3.3 cm.

A manufacturer wishes to make a cuboidal container with a lid that holds 12 tennis balls. The container is to be made of cardboard. The manufacturer wishes to use as little cardboard as possible.



1. Sketch some of the different containers that the manufacturer might consider.
2. For each container, calculate the total area of cardboard used and therefore decide on the most economical design.

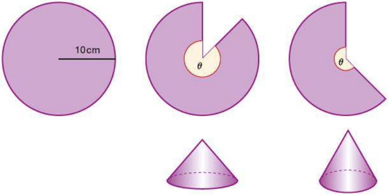
The manufacturer now considers the possibility of using other flat-faced containers.

3. Sketch some of the different containers that the manufacturer might consider.
4. Investigate the different amounts of cardboard used for each design.
5. Which type of container would you recommend to the manufacturer?

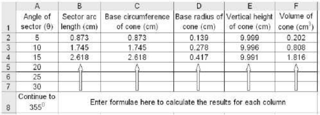
ICT Activity

In this topic you will have seen that it is possible to construct a cone from a sector. The dimensions of the cone are dependent on the dimensions of the sector. In this activity you will be using a spreadsheet to investigate the maximum possible volume of a cone constructed from a sector of fixed radius.

Circles of radius 10 cm are cut from paper and used to construct cones. Different sized sectors are cut from the circles and then arranged to form a cone, e.g.



1. Using a spreadsheet similar to the one below, calculate the maximum possible volume, for a cone constructed from one of these circles:



	A	B	C	D	E	F
1	Angle of sector (θ)	Sector arc length (cm)	Base circumference of cone (cm)	Base radius of cone (cm)	Vertical height of cone (cm)	Volume of cone (cm^3)
2	5	0.873	0.873	0.139	9.999	0.202
3	10	1.745	1.745	0.278	9.996	0.808
4	15	2.618	2.618	0.417	9.991	1.816
5	20	⇧	⇧	⇧	⇧	⇧
6	25					
7	30					
8	Continue to 355^0	Enter formulae here to calculate the results for each column				

2. Plot a graph to show how the volume changes as θ increases Comment on your graph.

SECTION 7

Student assessments

Student assessment 1

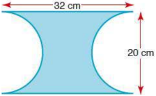
1. Calculate the circumference and area of each of the following circles. Give your answers to 1 d.p.

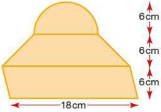
a)

b)

2. A rectangle of length 32 cm and width 20 cm has a semi-circle cut out of two of its sides as shown:

Calculate the shaded area to 1 d.p.





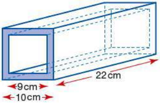
3. Calculate the area of:

a) the semi-circle

b) the parallelogram

c) the whole shape.

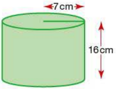
4. A prism in the shape of a hollowed-out cuboid has dimensions as shown below.



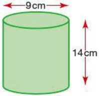
If the end is square, calculate the volume of the prism.

5. Calculate the surface area of each of the following cylinders:

a)



b)



Student assessment 2

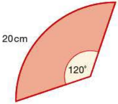
1. Calculate the arc length of the following sectors. The angle θ and radius r are given in both cases.

a) $\theta = 255°$ $r = 40\,\text{cm}$ b) $\theta = 240°$ $r = 16.3\,\text{mm}$

2. Calculate the angle θ in each of the following sectors. The radius r and arc length a are given in both cases.

a) $r = 40\,\text{cm}$ $a = 100\,\text{cm}$ b) $r = 20\,\text{cm}$ $a = 10\,\text{mm}$

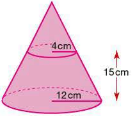
3. Calculate the area of the sector shown below:



4. A hemisphere has a radius of 8 cm. Calculate to 1 d.p.:

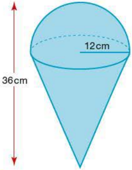
a) its total surface area
b) its volume.

5. A cone has its top cut as shown below.



Calculate:

a) the height of the large cone
b) the volume of the small cone
c) the volume of the truncated cone.



6. A metal object is made from a hemisphere and a cone, both of base radius 12 cm. The height of the object, when upright is 36 cm.

Calculate:

a) the volume of the hemisphere
b) the volume of the cone
c) the curved surface area of the hemisphere
d) the total surface area of the object.

TOPIC 7

Coordinate geometry

This topic will cover the following syllabus content:

7.1 Plotting of points and reading from a graph in the cartesian plane
7.2 Distance between two points
7.3 Midpoint of a line segment
7.4 Gradient of a line segment
7.5 Gradient of parallel and perpendicular lines
7.6 Equation of a straight line as $y = mx + c$ and $ax + by = d$ (a, b and d integer)
7.7 Linear inequalities on the cartesian plane (In Topic 2)
7.8 Symmetry of diagrams or graphs in the cartesian plane (In Topic 5)

Sections

SECTION
1

The French

Rene Descartes (1596–1650)

In the middle of the seventeenth century there were three great French mathematicians, Rene Descartes, Blaise Pascal and Pierre de Fermat.

Rene Descartes was a philosopher and a mathematician. His book *The Meditations* asks 'How and what do I know?' His work in mathematics made a link between algebra and geometry. He thought that all nature could be explained in terms of mathematics. Although he was not considered as talented a mathematician as Pascal and Fermat, he has had greater influence on modern thought. The (x, y) coordinates we use are called Cartesian coordinates after Descartes.

Blaise Pascal (1623–1662) was a genius who studied geometry as a child. When he was 16 he stated and proved Pascal's Theorem, which relates any six points on any conic section. The Theorem is sometimes called the 'Cat's Cradle'. He founded probability theory and made contributions to the invention of calculus. He is best known for Pascal's Triangle.

Pierre de Fermat (1601–1665) was a brilliant mathematician and, along with Descartes, one of the most influential. Fermat invented number theory and worked on calculus. He discovered probability theory with his friend Pascal. It can be argued that Fermat was at least Newton's equal as a mathematician.

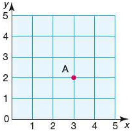
Fermat's most famous discovery in number theory includes 'Fermat's Last Theorem'. This theorem is derived from Pythagoras' theorem which states that for a right-angled triangle, $x^2 = y^2 + z^2$ where x is the length of the hypotenuse. Fermat said that if the index (power) was greater than two and x, y, z are all whole numbers, then the equation was never true. (This theorem was only proved in 1995 by the English mathematician Andrew Wiles.)

Coordinates

On 22 October 1707 four English war ships, *The Association* the flagship of Admiral Sir Clowdisley Shovell and three others, struck the Gilstone Ledges off the Scilly Isles and more than two thousand men drowned. Why? Because the Admiral had no way of knowing exactly where he was. He needed two coordinates to place his position on the sea. He only had one, his latitude.

The story of how to solve the problem of fixing the second coordinate (longitude) is told in Dava Sobel's book *Longitude*. The British Government offered a prize of £20 000 (millions of pounds at today's prices) to anyone who could solve the problem of how to fix longitude at sea.

Coordinates

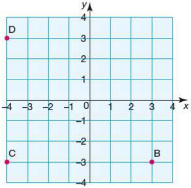


To fix a point in two dimensions (2D), its position is given in relation to a point called the **origin**. Through the origin, axes are drawn perpendicular to each other. The horizontal axis is known as the ***x*-axis**, and the vertical axis is known as the ***y*-axis**.

The *x*-axis is numbered from left to right. The *y*-axis is numbered from bottom to top.

The position of point A is given by two coordinates: the *x*-coordinate first, followed by the *y*-coordinate. So the coordinates of point A are (3, 2).

A number line can extend in both directions by extending the *x*- and *y*-axes below zero, as shown in the grid below:



Points B, C, and D can be described by their coordinates:

Point B is at (3, –3)
Point C is at (–4, –3)
Point D is at (–4, 3)

Exercise 7.1

1. Draw a pair of axes with both x and y from –8 to +8.
Mark each of the following coordinates on your grid:

a) A = (5, 2)
b) B = (7, 3)
c) C = (2, 4)
d) D = (–8, 5)
e) E = (–6, –8)
f) F = (3, –7)
g) G = (7, –3)
h) H = (6, –6)

Draw a separate grid for each of Q.2–4 with x- and y-axes from –6 to +6. Plot and join the point in order to name each shape drawn.

2. A = (3, 2) B = (3, –4) C = (–2, –4) D = (–2, 2)

3. E = (1, 3) F = (4, –5) G = (–2, –5)

4. H = (–6, 4) I = (0, –4) J = (4, –2) K = (–2, 6)

Exercise 7.2

Draw a pair of axes with both x and y from –10 to +10.

1. Plot the points P = (–6, 4), Q = (6, 4) and R = (8, –2).

Plot point S such that PQRS when drawn is a parallelogram.

a) Draw diagonals PR and QS. What are the coordinates of their point of intersection?
b) What is the area of PQRS?

2. On the same axes, plot point M at (–8, 4) and point N at (4, 4).
a) Join points MNRS. What shape is formed?
b) What is the area of MNRS?
c) Explain your answer to Q.2(b).

3. a) On the same axes, plot point J where point J has y-coordinate +10 and JRS, when joined, forms an isosceles triangle.
b) What is the x-coordinate of all points on the axis of symmetry of triangle JRS?

Exercise 7.3

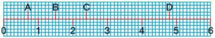
1. a) On a grid with axes numbered from –10 to +10 draw a regular hexagon ABCDEF with centre (0, 0) and coordinate A (0, 8).
b) Write down the approximate coordinates of points B, C, D, E and F.

2. a) On a similar grid to Q.1, draw an octagon PQRSTUVW which has point P (2, –8), point Q (–6, –8) and point R (–7, –5).
PQ = RS = TU = VW and QR = ST = UV = WP.
b) List the coordinates of points S, T, U, V, and W.
c) What is the coordinate of the centre of rotational symmetry of the octagon?

Exercise 7.4

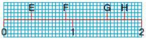
1. The points A, B, C and D are not at whole number points on the number line. Point A is at 0.7

   What is the position of points B, C and D?

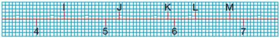


2. On this number line point E is at 0.4 (2 small squares represents 0.1)

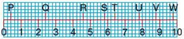
   What is the position of points F, G and H.



3. What is the position of points I, J, K, L and M? (Each small square is 0.05, i.e 2 squares is 0.1)

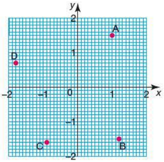


4. Point P is at position 0.4 and point W is at position 9.8 (Each small square is 0.2)
   What is the position of points Q, R, S, T, U, and V?

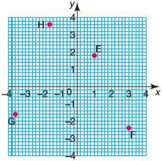


Exercise 7.5

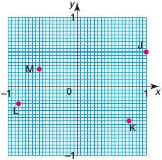
1. Give the coordinates of points A, B, C and D.



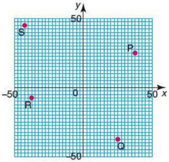
2. Give the coordinates of points E, F, G, H.



3. Give the coordinates of points J, K, L and M.



4. Give the coordinates of points P, Q, R and S.

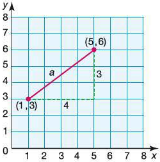


SECTION 3 Line segments

Calculating the length of a line segment

A line segment is formed when two points are joined by a straight line. To calculate the distance between two points, and therefore the length of the line segment, their coordinates need to be given. Once these are known, Pythagoras' theorem can be used to calculate the distance.

Worked example The coordinates of two points are (1, 3) and (5, 6). Draw a pair of axes, plot the given points and calculate the distance between them.



By dropping a vertical line from the point (5, 6) and drawing a horizontal line from (1, 3), a right-angled triangle is formed. The length of the hypotenuse of the triangle is the length we wish to find.

Using Pythagoras' theorem, we have:

$$a^2 = 3^2 + 4^2$$
$$a^2 = 25$$
$$a = \sqrt{25}$$
$$a = 5$$

The length of the line segment is 5 units.

To find the distance between two points directly from their coordinates, use the following formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)}^2$$

Worked example Without plotting the points, calculate the distance between the points (1, 3) and (5, 6).

$$\begin{aligned} d &= \sqrt{(1-5)^2 + (3-6)^2} \\ &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

The distance between the two points is 5 units.

The midpoint of a line segment

To find the midpoint of a line segment, use the coordinates of its end points. To find the x-coordinate of the midpoint, find the mean of the x-coordinates of the end points. Similarly, to find the y-coordinate of the midpoint, find the mean of the y-coordinates of the end points.

Worked examples

a) Find the coordinates of the midpoint of the line segment AB where A is (1, 3) and B is (5, 6).

The x-coordinate of the midpoint will be $\frac{1+5}{2} = 3$

The y-coordinate of the midpoint will be $\frac{3+6}{2} = 4.5$

So the coordinates of the midpoint are (3, 4.5)

b) Find the coordinates of the midpoint of a line segment PQ where P is (–2, –5) and Q is (4, 7).

The x-coordinate of the midpoint will be $\frac{-2+4}{2} = 1$

The y-coordinate of the midpoint will be $\frac{-5+7}{2} = 1$

So the coordinates of the midpoint are (1, 1).

Exercise 7.6

1. i) Plot each of the following pairs of points.
ii) Calculate the distance between each pair of points.
iii) Find the coordinates of the midpoint of the line segment joining the two points.

a) (5, 6) (1, 2)
b) (6, 4) (3, 1)
c) (1, 4) (5, 8)
d) (0, 0) (4, 8)
e) (2, 1) (4, 7)
f) (0, 7) (–3, 1)
g) (–3, –3) (–1, 5)
h) (4, 2) (–4, –2)
i) (–3, 5) (4, 5)
j) (2, 0) (2, 6)
k) (–4, 3) (4, 5)
l) (3, 6) (–3, –3)

2. Without plotting the points:
i) calculate the distance between each of the following pairs of points
ii) find the coordinates of the midpoint of the line segment joining the two points.

a) (1, 4) (4, 1)
b) (3, 6) (7, 2)
c) (2, 6) (6, –2)
d) (1, 2) (9, –2)
e) (0, 3) (–3, 6)
f) (–3, –5) (–5, –1)
g) (–2, 6) (2, 0)
h) (2, –3) (8, 1)
i) (6, 1) (–6, 4)
j) (–2, 2) (4, –4)
k) (–5, –3) (6, –3)
l) (3, 6) (5, –2)

Gradient of a straight line

The **gradient** of a straight line refers to its 'steepness' or 'slope'. The gradient of a straight line is constant, i.e. it does not change. The gradient can be calculated by considering the coordinates of any two points (x_1, y_1), (x_2, y_2) on the line. It is calculated using the following formula:

$$\text{Gradient} = \frac{\text{vertical distance between the two points}}{\text{horizontal distance between the two points}}$$

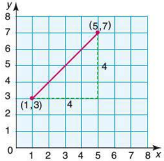
By considering the x- and y-coordinates of the two points, this can be rewritten as:

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Worked examples

a) The coordinates of two points on a straight line are (1, 3) and (5, 7). Plot the two points on a pair of axes and calculate the gradient of the line joining them.

$$\text{Gradient} = \frac{7-3}{5-1} = \frac{4}{4} = 1$$

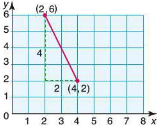


Note: It does not matter which point we choose to be (x_1, y_1) or (x_2, y_2) as the gradient will be the same. In the example above, reversing the points:

$$\text{Gradient} = \frac{3-7}{1-5} = \frac{-4}{-4} = 1$$

b) The coordinates of two points on a straight line are (2, 6) and (4, 2). Plot the two points on a pair of axes and calculate the gradient of the line joining them.

$$\text{Gradient} = \frac{2-6}{4-2} = \frac{-4}{2} = -2$$

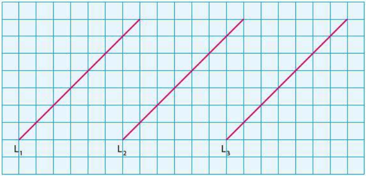


To check whether or not the sign of the gradient is correct, the following guideline is useful:

A line sloping this way will have a positive gradient

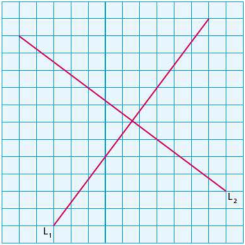
A line sloping this way will have a negative gradient

Parallel lines will have the same gradient. Conversely, lines which have the same gradient are parallel. If two lines are parallel to each other their gradients m_1 and m_2 are equal.



L_1, L_2, L_3 all have the same gradient so are parallel.

The *x*-axis and the *y*-axis on a graph intersect at right angles. They are perpendicular to each other. In the graph below, L_1 and L_2 are perpendicular to each other.



The gradient m_1 of line L_1 is $\frac{4}{3}$ whilst the gradient m_2 of line L_2 is $-\frac{3}{4}$.

The product of m_1m_2 gives the result -1, i.e. $\frac{4}{3} \times (-\frac{3}{4}) = -1$

If two lines are perpendicular to each other, the product of their gradients is -1, i.e. $m_1m_2 = -1$.

Therefore the gradient of one line is the negative reciprocal of the other line, i.e. $m_1 = \frac{-1}{m_2}$.

Exercise 7.7

1. With the aid of axes if necessary, calculate:

i) the gradient of the line joining the following pairs of points
ii) the gradient of a line perpendicular to this line.

a) (5, 6) (1, 2)
b) (6, 4) (3, 1)
c) (1, 4) (5, 8)
d) (0, 0) (4, 8)
e) (2, 1) (4, 7)
f) (0, 7) (−3, 1)
g) (−3, −3) (−1, 5)
h) (4, 2) (−4, −2)
i) (−3, 5) (4, 5)
j) (2, 0) (2, 6)
k) (−4, 3) (4, 5)
l) (3, 6) (−3, −3)

2. With the aid of axes if necessary, calculate:

i) the gradient of the line joining the following pairs of points
ii) the gradient of a line perpendicular to this line.

a) (1, 4) (4, 1)
b) (3, 6) (7, 2)
c) (2, 6) (6, −2)
d) (1, 2) (9, −2)
e) (0, 3) (−3, 6)
f) (−3, −5) (−5, −1)
g) (−2, 6) (2, 0)
h) (2, −3) (8, 1)
i) (6, 1) (−6, 4)
j) (−2, 2) (4, −4)
k) (−5, −3) (6, −3)
l) (3, 6) (5, −2)

SECTION 4

Equation of a straight line

The coordinates of every point on a straight line all have a common relationship. This relationship, when expressed algebraically as an equation in terms of *x* and/or *y*, is known as the equation of the straight line.

Worked examples

a) By looking at the coordinates of some of the points on the line below, establish the equation of the straight line.

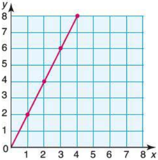
x	y
1	4
2	4
3	4
4	4
5	4
6	4

Some of the points on the line have been identified and their coordinates entered in the table above. By looking at the table, it can be seen that the only rule all the points have in common is that $y = 4$.

Hence the equation of the straight line is $y = 4$.

b) By looking at the coordinates of some of the points on the line, establish the equation of the straight line.

x	y
1	2
2	4
3	6
4	8



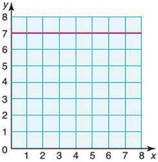
Once again, by looking at the table it can be seen that the relationship between the *x*- and *y*-coordinates is that each *y*-coordinate is twice the corresponding *x*-coordinate.

Hence the equation of the straight line is $y = 2x$.

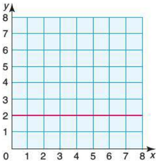
Exercise 7.8

For each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

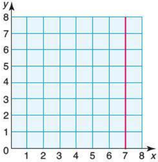
1.



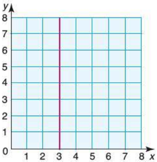
2.



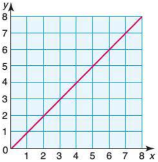
2.



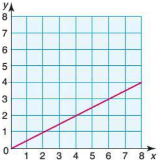
3.



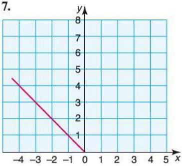
5.



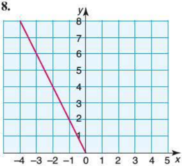
6.



7.



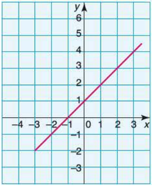
8.



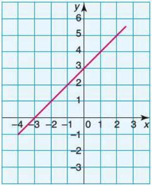
Exercise 7.9

1 For each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

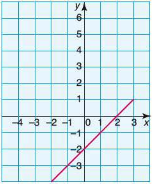
a)



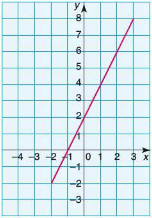
b)



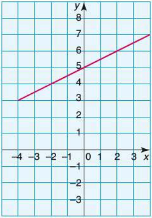
c)



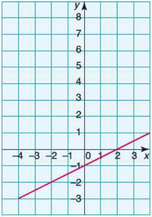
d)



e)

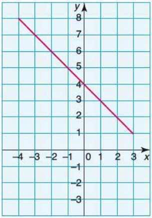


f)

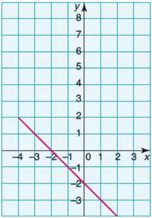


2. For each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

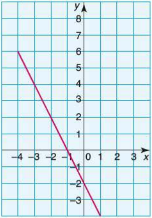
a)



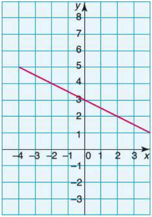
b)



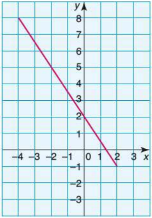
c)



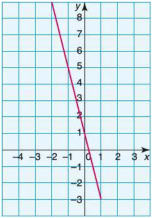
d)



e)



f)



3. a) For each of the graphs in Q.1 and 2, calculate the gradient of the straight line.

b) What do you notice about the gradient of each line and its equation?

c) What do you notice about the equation of the straight line and where the line intersects the y-axis?

4. Copy the diagrams in Q.1. Draw two lines on the diagram parallel to the given line.
   a) Write the equation of these new lines in the form $y = mx + c$.
   b) What do you notice about the equations of these new parallel lines?

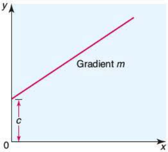
5. In Q.2 you found an equation for these lines in the form $y = mx + c$. Change the value of the intercept c and then draw the new line.
   What do you notice about this new line and the first line?

In general the equation of any straight line can be written in the form:

$$y = mx + c$$

where m represents the gradient of the straight line and c the intercept with the y-axis. This is shown in the diagram.

By looking at the equation of a straight line written in the form $y = mx + c$, it is therefore possible to deduce the line's gradient and intercept with the y-axis without having to draw it.



Worked examples

a) Find the gradient and y-intercept of the following straight lines:

i) $y = 3x - 2$ gradient = 3
y-intercept = –2

ii) $y = -2x + 6$ gradient = –2
y-intercept = 6

b) Calculate the gradient and y-intercept of the following straight lines:

i) $2y = 4x + 2$

This needs to be rearranged into **gradient–intercept** form (i.e. $y = mx + c$):

$y = 2x + 1$ gradient = 2
y-intercept = 1

ii) $y - 2x = -4$

Rearranging into gradient–intercept form:

$y = 2x - 4$ gradient = 2
y-intercept = –4

iii) $-4y + 2x = 4$

Rearranging into gradient–intercept form:

$y = \frac{1}{2}x - 1$ gradient $= \frac{1}{2}$
y-intercept $= -1$

iv) $\frac{y+3}{4} = -x + 2$

Rearranging into gradient–intercept form:

$y + 3 = -4x + 8$
$y = -4x + 5$ gradient $= -4$
y-intercept $= 5$

Exercise 7.10

For the following linear equations, calculate both the gradient and y-intercept.

1. a) $y = 2x + 1$ b) $y = 3x + 5$ c) $y = x - 2$
d) $y = \frac{1}{2}x + 4$ e) $y = -3x + 6$ f) $y = -\frac{2}{3}x + 1$
g) $y = -x$ h) $y = -x - 2$ i) $y = -(2x - 2)$

2. a) $y - 3x = 1$ b) $y + \frac{1}{2}x - 2 = 0$ c) $y + 3 = -2x$
d) $y + 2x + 4 = 0$ e) $y - \frac{1}{4}x - 6 = 0$ f) $-3x + y = 2$
g) $2 + y = x$ h) $8x - 6 + y = 0$ i) $-(3x + 1) + y = 0$

3. a) $2y = 4x - 6$ b) $2y = x + 8$ c) $\frac{1}{2}y = x - 2$
d) $\frac{1}{4}y = -2x + 3$ e) $3y - 6x = 0$ f) $\frac{1}{3}y + x = 1$
g) $6y - 6 = 12x$ h) $4y - 8 + 2x = 0$ i) $2y - (4x - 1) = 0$

4. a) $2x - y = 4$ b) $x - y + 6 = 0$ c) $-2y = 6x + 2$
d) $12 - 3y = 3x$ e) $5x - \frac{1}{2}y = 1$ f) $-\frac{2}{3}y + 1 = 2x$
g) $9x - 2 = -y$ h) $-3x + 7 = -\frac{1}{2}y$ i) $-(4x - 3) = -2y$

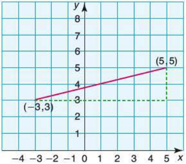
5. a) $\frac{y+2}{4} = \frac{1}{2}x$ b) $\frac{y-3}{x} = 2$ c) $\frac{y-x}{8} = 0$
d) $\frac{2y-3x}{2} = 6$ e) $\frac{3y-2}{x} = -3$ f) $\frac{\frac{1}{2}y - 1}{x} = -2$
g) $\frac{3x-y}{2} = 6$ h) $\frac{6-2y}{3} = 2$ i) $\frac{-(x+2y)}{5x} = 1$

6. a) $\frac{3x-y}{y} = 2$ b) $\frac{-x+2y}{4} = y + 1$
c) $\frac{y-x}{x+y} = 2$ d) $\frac{1}{y} = \frac{1}{x}$
e) $\frac{-(6x+y)}{2} = y + 1$ f) $\frac{2x-3y+4}{4} = 4$
g) $\frac{y+1}{x} + \frac{3y-2}{2x} = -1$ h) $\frac{x}{y+1} + \frac{1}{2y+2} = -3$
i) $\frac{-(-y+3x)}{-(6x-2y)} = 1$ j) $\frac{-(x-3y)-(-x-2y)}{4+x-y} = -2$

The equation of a line through two points

The equation of a straight line can be found once the coordinates of two points on the line are known.

Worked example Calculate the equation of the straight line passing through the points (–3, 3) and (5, 5). Plotting the two points gives:



The equation of any straight line can be written in the general form $y = mx + c$. Here we have:

$$\text{Gradient} = \frac{5-3}{5-(-3)} = \frac{2}{8} = \frac{1}{4}$$

The equation of the line now takes the form $y = \frac{1}{4}x + c$.

Since the line passes through the two given points, their coordinates must satisfy the equation. So to calculate the value of c, the x and y coordinates of one of the points are substituted into the equation. Substituting (5, 5) into the equation gives:

$$5 = \tfrac{1}{4} \times 5 + c$$
$$5 = 1\tfrac{1}{4} + c$$

Therefore $c = 5 - 1\frac{1}{4} = 3\frac{3}{4}$

The equation of the straight line passing through (–3, 3) and (5, 5) is:

$$y = \tfrac{1}{4}x + 3\tfrac{3}{4}$$

We have seen that the equation of a straight line takes the form $y = mx + c$. It can, however, also take the form $ax + by = d$. It is possible to write the equation $y = mx + c$ in the form $ax + by = d$ by rearranging the equation.

In the example above, $y = \frac{1}{4}x + 3\frac{3}{4}$ can firstly be rewritten as:

$$y = \frac{x}{4} + \frac{15}{4}$$

Multiplying both sides of the equation by 4 produces the equation $4y = x + 15$.

This can be rearranged to $-x + 4y = 15$, which is the required form with $a = -1$, $b = 4$ and $d = 15$.

Exercise 7.11

Find the equation of the straight line which passes through each of the following pairs of points. Express your answers in the form:

i) $y = mx + c$
ii) $ax + by = d$.

1. a) $(1, 1)$ $(4, 7)$
b) $(1, 4)$ $(3, 10)$
c) $(1, 5)$ $(2, 7)$
d) $(0, -4)$ $(3, -1)$
e) $(1, 6)$ $(2, 10)$
f) $(0, 4)$ $(1, 3)$
g) $(3, -4)$ $(10, -18)$
h) $(0, -1)$ $(1, -4)$
i) $(0, 0)$ $(10, 5)$

2. a) $(-5, 3)$ $(2, 4)$
b) $(-3, -2)$ $(4, 4)$
c) $(-7, -3)$ $(-1, 6)$
d) $(2, 5)$ $(1, -4)$
e) $(-3, 4)$ $(5, 0)$
f) $(6, 4)$ $(-7, 7)$
g) $(-5, 2)$ $(6, 2)$
h) $(1, -3)$ $(-2, 6)$
i) $(6, -4)$ $(6, 6)$

Drawing straight line graphs

To draw a straight line graph only two points need to be known. Once these have been plotted the line can be drawn between them and extended if necessary at both ends. It is important to check your line is correct by taking a point from your graph and ensuring it satisfies the original equation.

Worked examples

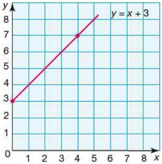
a) Plot the line $y = x + 3$.

To identify two points simply choose two values of x. Substitute these into the equation and calculate their corresponding y-values.

When $x = 0$, $y = 3$.
When $x = 4$, $y = 7$.

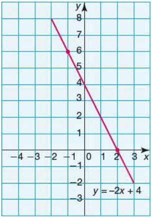
Therefore two of the points on the line are $(0, 3)$ and $(4, 7)$.

The straight line $y = x + 3$ is plotted as shown.



Check using a third point, e.g. $(1, 4)$.

When $x = 1$, $y = x + 3 = 4$, so $(1, 4)$ satisfies the equation of the line.



b) Plot the line $y = -2x + 4$.

When $x = 2$, $y = 0$.
When $x = -1$, $y = 6$.

The coordinates of two points on the line are $(2, 0)$ and $(-1, 6)$, and the line is plotted as shown.

Check using the point $(0, 4)$.

When $x = 0$, $y = -2x + 4 = 4$, so $(0, 4)$ satisfies the equation of the line.

Note that, in questions of this sort, it is often easier to rearrange the equation into gradient-intercept form first.

Exercise 7.12

1. Plot the following straight lines:

a) $y = 2x + 3$
b) $y = x - 4$
c) $y = 3x - 2$
d) $y = -2x$
e) $y = -x - 1$
f) $-y = \frac{1}{2}x + 4$
g) $-y = 3x - 3$
h) $2y = 4x - 2$
i) $y - 4 = 3x$

2. Plot the following straight lines:

a) $-2x + y = 4$
b) $-4x + 2y = 12$
c) $3y = 6x - 3$
d) $2x = x + 1$
e) $3y - 6x = 9$
f) $2y + x = 8$
g) $x + y + 2 = 0$
h) $3x + 2y - 4 = 0$
i) $4 = 4y - 2x$

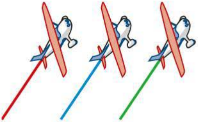
3. Plot the following straight lines:

a) $\frac{x + y}{2} = 1$
b) $x + \frac{y}{2} = 1$
c) $\frac{x}{3} + \frac{y}{2} = 1$
d) $y + \frac{x}{2} = 3$
e) $\frac{y}{5} + \frac{x}{3} = 0$
f) $\frac{-(2x + y)}{4} = 1$
g) $\frac{y - (x - y)}{3x} = -1$
h) $\frac{y}{2x + 3} - \frac{1}{2} = 0$
i) $-2(x + y) + 4 = -y$

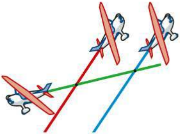
Plane trails

In an aircraft show, planes are made to fly with a coloured smoke trail. Depending on the formation of the planes, the trails can intersect in different ways.

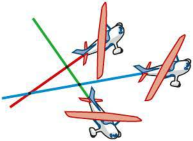
In the diagram below the three smoke trails do not cross as they are parallel.



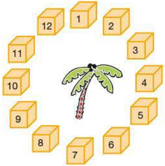
In the following diagram there are two crossing points.



By flying differently, the three planes can produce trails that cross at three points.



1. Investigate the connection between the maximum number of crossing points and the number of planes.
2. Record the results of your investigation in an ordered table.
3. Write an algebraic rule linking the number of planes (p) and the maximum number of crossing points (n).



Hidden treasure

A television show sets up a puzzle for its contestants to try and solve. Some buried treasure is hidden on a 'treasure island'. The treasure is hidden in one of the 12 treasure chests shown (left). Each contestant stands by one of the treasure chests.

The treasure is hidden according to the following rule:

> It is not hidden in chest 1.
> Chest 2 is left empty for the time being.
> It is not hidden in chest 3.
> Chest 4 is left empty for the time being.
> It is not hidden in chest 5.
> The pattern of crossing out the first chest and then alternate chests is continued until only one chest is left. This will involve going round the circle several times continuing the pattern.
> The treasure is hidden in the last chest left.

The diagrams below show how the last chest is chosen:

After 1 round, chests 1, 3, 5, 7, 9 and 11 have been discounted.

After the second round, chests 2, 6 and 10 have also been discounted.

After the third round, chests 4 and 12 have also been discounted. This leaves only chest 8.
The treasure is therefore hidden in chest 8.

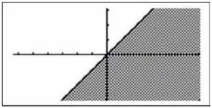
Unfortunately for participants, the number of contestants changes each time.

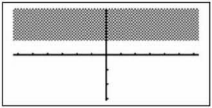
1. Investigate which treasure chest you would choose if there are:
   a) 4 contestants
   b) 5 contestants
   c) 8 contestants
   d) 9 contestants
   e) 15 contestants.
2. Investigate the winning treasure chest for other numbers of contestants and enter your results in an ordered table.

3. State any patterns you notice in your table of results.
4. Use your patterns to predict the winning chest for 31, 32 and 33 contestants.
5. Write a rule linking the winning chest x and the number of contestants n.

ICT Activity

Your graphics calculator is able to graph inequalities and shade the appropriate region. The examples below show some screenshots taken from a graphics calculator.





Investigate how your calculator can graph linear inequalities.

SECTION 6

Student assessments

Student assessment 1

1. Sketch the following graphs on the same pair of axes, labelling each clearly.
   a) $x = 3$ b) $y = -2$
   c) $y = -3x$ d) $y = \frac{x}{4} + 4$
2. For each of the following linear equations:
   i) calculate the gradient and y-intercept
   ii) plot the graph.
   a) $y = 2x + 3$ b) $y = 4 - x$
   c) $2x - y = 3$ d) $-3x + 2y = 5$
3. Find the equation of the straight line which passes through the following pairs of points:
   a) $(-2, -9)$ $(5, 5)$ b) $(1, -1)$ $(-1, 7)$
4. The coordinates of the end points of two line segments are given below. Calculate the length of each of the lines.
   a) $(2, 6)$ $(-2, 3)$ b) $(-10, -10)$ $(0, 14)$

TOPIC
8

Trigonometry

This topic will cover the following syllabus content:

8.1 Right-angled triangle trigonometry
8.2 Exact values for the trig ratios of 0°, 30°, 45°, 60°, 90°
8.3 Extension to the four quadrants, i.e. 0–360°
8.4 Sine rule
8.5 Cosine rule
8.6 Area of triangle
8.7 Applications: three-figure bearings and North, East, South, West
problems in two and three dimensions
compound shapes
8.8 Properties of the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$

Sections

SECTION 1 The Swiss

Leonhard Euler

Leonhard Euler (1707–1783)

Euler, like Newton, was the greatest mathematician of his generation. He studied all areas of mathematics and continued to work hard after he had gone blind.

As a young man, Euler discovered and proved:

the sum of the infinite series $\sum\left(\frac{1}{n^2}\right) = \frac{\pi^2}{6}$

i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6}$

This brought him to the attention of other mathematicians.

Euler made discoveries in many areas of mathematics, especially calculus and trigonometry. He also developed the ideas of Newton and Leibniz.

Euler worked on graph theory and functions and was the first to prove several theorems in geometry. He studied relationships between a triangle's height, mid-point, and circumscribing and inscribing circles, and an expression for a tetrahedron's (a triangular pyramid) area in terms of its sides.

He also worked on number theory and found the largest prime number known at the time.

Some of the most important constant symbols in mathematics, π, e and i (the square root of –1), were introduced by Euler.

The Bernoulli family

The Bernoullis were a family of Swiss merchants who were friends of Euler. The two brothers, Johann and Jacob, were very gifted mathematicians and scientists, as were their children and grandchildren. They made discoveries in calculus, trigonometry and probability theory in mathematics. In science, they worked on astronomy, magnetism, mechanics, thermodynamics and more.

Unfortunately many members of the Bernouilli family were not pleasant people. The older members of the family were jealous of each other's successes and often stole the work of their sons and grandsons and pretended that it was their own.

SECTION 2

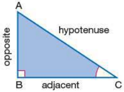
Sine, cosine and tangent ratios

NB: Diagrams are not drawn to scale.

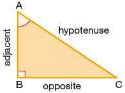
There are three basic trigonometric ratios: sine, cosine and tangent.

Each of these relates an angle of a right-angled triangle to a ratio of the lengths of two of its sides.

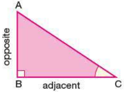
The sides of the triangle have names, two of which are dependent on their position in relation to a specific angle. The longest side (always opposite the right angle) is called the **hypotenuse**. The side opposite the angle is called the **opposite** side and the side next to the angle is called the **adjacent** side.



Note that, when the chosen angle is at A, the sides labelled opposite and adjacent change as shown:



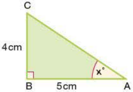
Tangent



$$\tan C = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

Worked examples

a) Calculate the size of angle BAC in the following triangles:

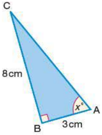


i) $\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{5}$

$x = \tan^{-1}\left(\frac{4}{5}\right)$

$x = 38.7$ (1 d.p.)

$\angle BAC = 38.7°$ (1 d.p.)

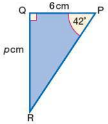


ii) $\tan x° = \frac{8}{3}$

$x = \tan^{-1}\left(\frac{8}{3}\right)$

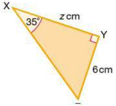
$x = 69.4$ (1 d.p.)

$\angle BAC = 69.4°$ (1 d.p.)



b) Calculate the length of the opposite side QR.

$$\tan 42° = \frac{p}{6}$$
$$6 \times \tan 42° = p$$
$$p = 5.40 \text{ (3 s.f.)}$$
$$\text{QR} = 5.40 \text{ cm (3 s.f.)}$$



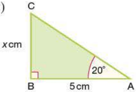
c) Calculate the length of the adjacent side XY.

$$\tan 35° = \frac{6}{z}$$
$$z \times \tan 35° = 6$$
$$z = \frac{6}{\tan 35°}$$
$$z = 8.57 \text{ (3 s.f.)}$$
$$\text{XY} = 8.57 \text{ cm (3 s.f.)}$$

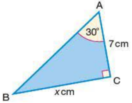
Exercise 8.1

Calculate the length of the side marked x cm in each of the diagrams in Q. 1 and 2. Give your answers to 3 s.f.

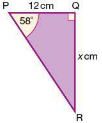
1. a)



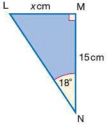
b)



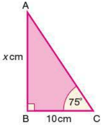
c)



d)



e)



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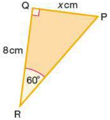
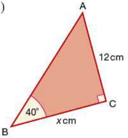
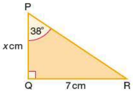


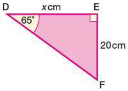
2. a)



b)



c)



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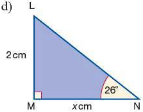

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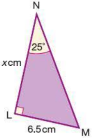

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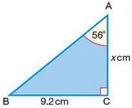
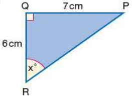
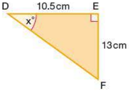


3. Calculate the size of the marked angle $x°$ in each of the following diagrams. Give your answer to 1 d.p.

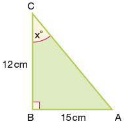
a)



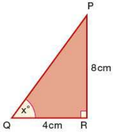
b)



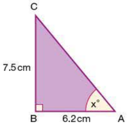
c)



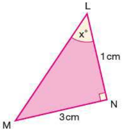
d)



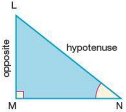
e)



f)

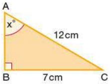


Sine



$$\sin \text{N} = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

Worked examples



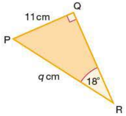
a) Calculate the size of angle BAC.

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{12}$$

$$x = \sin^{-1}\left(\frac{7}{12}\right)$$

$$x = 35.7 \text{ (1 d.p.)}$$

$$\angle \text{BAC} = 35.7° \text{ (1 d.p.)}$$



b) Calculate the length of the hypotenuse PR.

$$\sin 18° = \frac{11}{q}$$

$$q \times \sin 18° = 11$$

$$q = \frac{11}{\sin 18°}$$

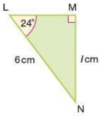
$$q = 35.6 \text{ (3 s.f.)}$$

$$\text{PR} = 35.6 \text{ cm (3 s.f.)}$$

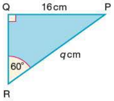
Exercise 8.2

1. Calculate the length of the marked side in each of the following diagrams. Give your answers to 3 s.f.

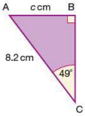
a)



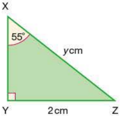
b)



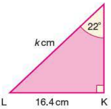
c)



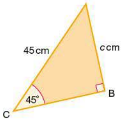
d)



e)

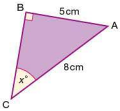


f)

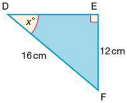


2. Calculate the size of the angle marked x in each of the following diagrams. Give your answers to 1 d.p.

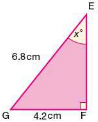
a)



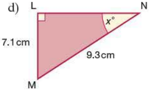
b)



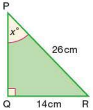
c)



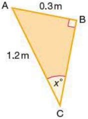
d)



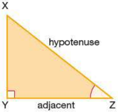
e)



f)



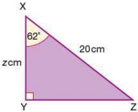
Cosine



$$\cos Z = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

Worked examples

a) Calculate the length XY.



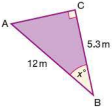
$$\cos 62^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{z}{20}$$

$$z = 20 \times \cos 62^\circ$$

$$z = 9.39 \text{ (3 s.f.)}$$

$$\text{XY} = 9.39 \text{ cm (3 s.f.)}$$

b) Calculate the size of angle ABC.



$$\cos x = \frac{5.3}{12}$$

$$x = \cos^{-1}\left(\frac{5.3}{12}\right)$$

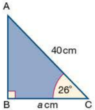
$$x = 63.8 \text{ (1 d.p.)}$$

$$\angle\text{ABC} = 63.8^\circ \text{ (1 d.p.)}$$

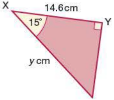
Exercise 8.3

Calculate either the marked side or angle in each of the following diagrams. Give your answers to 1 d.p.

1.



2.



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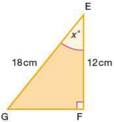

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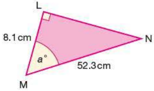

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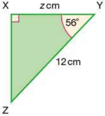

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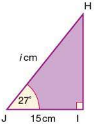

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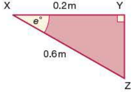

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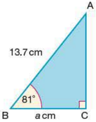
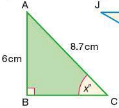


Exercise 8.4

1. By using Pythagoras' theorem, trigonometry or both, calculate the marked value in each of the following diagrams. In each case give your answer to 1 d.p.

Note: Pythagoras' theorem was covered in Section 3 of Topic 4.

a)



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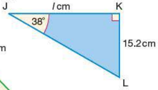
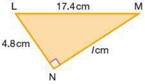
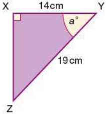
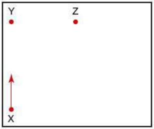


c)

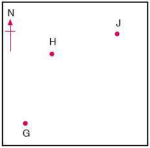


d)

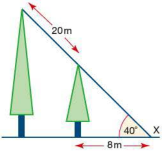




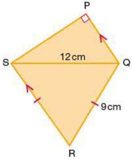
2. A sailing boat sets off from a point X and heads towards Y, a point 17 km North. At point Y it changes direction and heads towards point Z, a point 12 km away on a bearing of 090°. Once at Z the crew want to sail back to X. Calculate:
   a) the distance ZX
   b) the bearing of X from Z.



3. An aeroplane sets off from G on a bearing of 024° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 055° and a distance of 180 km away.
   a) How far is H to the North of G?
   b) How far is H to the East of G?
   c) How far is J to the North of H?
   d) How far is J to the East of H?
   e) What is the shortest distance between G and J?
   f) What is the bearing of G from J?



4. Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°. If the horizontal distance between X and the small tree is 8 m and the distance between the tops of the two trees is 20 m, calculate:
   a) the height of the small tree
   b) the height of the tall tree
   c) the horizontal distance between the trees.



5. PQRS is a quadrilateral. The sides RS and QR are the same length. The sides QP and RS are parallel.
Calculate:
   a) angle SQR
   b) angle PSQ
   c) length PQ
   d) length PS
   e) the area of PQRS.

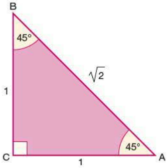
SECTION 3

Special angles and their trigonometric ratios

So far most of the angles you have worked with have required the use of a calculator in order to calculate their sine, cosine or tangent. However some angles produce exact values and therefore a calculator is both unnecessary and unhelpful when exact solutions are required.

There are a number of angles which have 'neat' trigonometric ratios, for example 0°, 30°, 45°, 60° and 90°. Their trigonometric ratios are derived below.

Consider the right-angled isosceles triangle ABC (left).



Let the perpendicular sides AC and BC each have a length of 1 unit.

As ΔABC is isosceles, $\angle ABC = \angle CAB = 45°$.

Using Pythagoras' theorem, AB can also be calculated:

$$AB^2 = AC^2 + BC^2$$
$$AB^2 = 1^2 + 1^2 = 2$$
$$AB = \sqrt{2}$$

From the triangle, it can be deduced that $\sin 45° = \frac{1}{\sqrt{2}}$

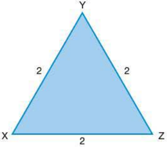
which when rationalised can be written as $\sin 45° = \frac{\sqrt{2}}{2}$

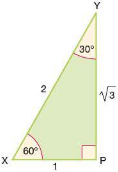
Similarly $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Therefore $\sin 45° = \cos 45°$

$\tan 45° = \frac{1}{1} = 1$

Consider also the equilateral triangle XYZ below, in which each of its sides have a length of 2 units.





If a vertical line is dropped from the vertex Y until it meets the base XY at P, the triangle is bisected. Consider now the right-angled triangle XYP.

$\angle XYP = 30°$ as it is half of $\angle XYZ$.
XP = 1 unit length as it is half of XZ.
The length YP can be calculated using Pythagoras' theorem:

$$XY^2 = XP^2 + YP^2$$
$$YP^2 = XY^2 - XP^2 = 2^2 - 1^2 = 3$$
$$YP = \sqrt{3}$$

Therefore from this triangle the following trigonometric ratios can be deduced:

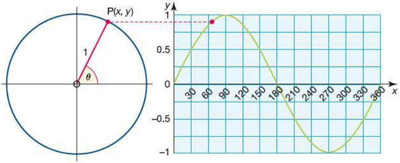
$$\sin 30° = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{1}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

These results and those obtained from the trigonometric graphs shown on the next page are summarised in the table below:

Angle (θ)	sin (θ)	cos (θ)	tan (θ)
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	–

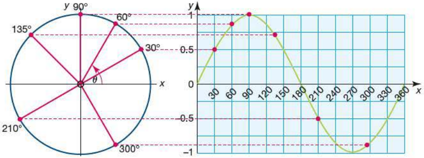
Other angles have the same trigonometric ratios as those shown on page 356. The following section explains why, using a unit circle, i.e. a circle with a radius of 1 unit.



In the diagram above, P is a point on the circumference of a circle with centre at O and a radius of 1 unit. P has coordinates (x, y). As the position of P changes, then so does the angle θ.

$\sin\theta = \frac{y}{1} = y$ i.e. the sine of the angle θ is represented by the y-coordinate of P.

The graph therefore shows the different values of $\sin\theta$ as θ varies. A more complete diagram is shown below. Note that the angle θ is measured anticlockwise from the positive x-axis.

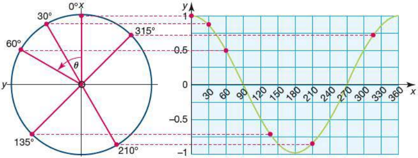


The graph of $y = \sin x$ has:

- a period of 360° (i.e. it repeats itself every 360°)
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\sin x = \sin(180 - x)$.

Similar diagrams and graphs can be constructed for $\cos\theta$ and $\tan\theta$.

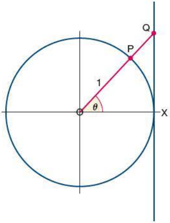
From the unit circle, it can be deduced that $\cos\theta = \frac{x}{1} = x$, i.e. the cosine of the angle θ is represented by the x-coordinate of P. Since $\cos\theta = x$, to be able to compare the graphs, the axes should be rotated through 90° as shown on the next page.



The properties of the cosine curve are similar to those of the sine curve. It has:

- a period of 360°
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\cos x = \cos(360 - x)$

The cosine curve is a translation of the sine curve of –90°, i.e $\sin x = \cos(x - 90)$



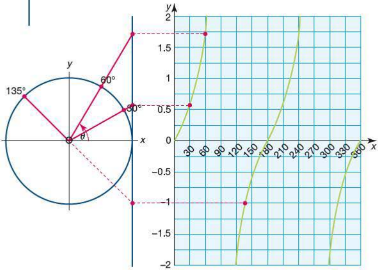
From the unit circle it can be deduced that $\tan\theta = \frac{y}{x}$.

In order to compare all the graphs a tangent to the unit circle is drawn at (1, 0). OP is extended to meet the tangent at Q as shown.

As OX = 1 (radius of the unit circle), $\tan\theta = \frac{\text{QX}}{\text{OX}} = \text{QX}$.

i.e. tan θ is equal to the y-coordinate of Q.

The graph of tan θ is therefore shown below:

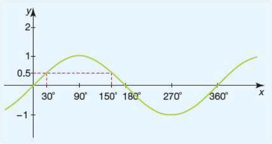


The graph of tan θ has:

- a period of 180°
- no maximum or minimum value
- symmetry
- asymptotes at 90° and 270°

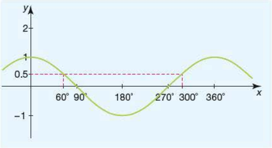
Worked examples

a) sin 30° = 0.5. Which other angle between 0° and 360° has a sine of 0.5?

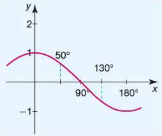


From the graph above it can be seen that sin 150° = 0.5. Also sin x = sin (180 − x), therefore sin 30° = sin (180 − 30) = sin 150°.

b) cos 60° = 0.5. Which other angle between 0° and 360° has a cosine of 0.5?



From the graph above it can be seen that cos 300° = 0.5.



c) The cosine of which angle between 0° and 180° is equal to the negative of cos 50°?

Cos 50° has the same magnitude but different sign to cos 130° due to the symmetrical properties of the cosine curve.

Therefore cos 130° = –cos 50°

Exercise 8.5

1. Express each of the following in terms of the sine of another angle between 0° and 360°:

a) sin 60° b) sin 80° c) sin 115°
d) sin 200° e) sin 300° f) sin 265°

2. Express each of the following in terms of the sine of another angle between 0° and 360°:

a) sin 35° b) sin 50° c) sin 30°
d) sin 248° e) sin 304° f) sin 327°

3. Find the two angles between 0° and 360° which have the following sine. Give each angle to the nearest degree.

a) 0.33 b) 0.99 c) 0.09
d) $-\frac{1}{2}$ e) $-\frac{\sqrt{3}}{2}$ f) $-\frac{1}{\sqrt{2}}$

4. Find the two angles between 0° and 360° which have the following sine. Give each angle to the nearest degree.

a) 0.94 b) 0.16 c) 0.80
d) –0.56 e) –0.28 f) –0.33

Exercise 8.6

1. Express each of the following in terms of the cosine of another angle between 0° and 360°:

a) cos 20° b) cos 85° c) cos 32°
d) cos 95° e) cos 147° f) cos 106°

2. Express each of the following in terms of the cosine of another angle between 0° and 360°:

a) cos 98° b) cos 144° c) cos 160°
d) cos 183° e) cos 211° f) cos 234°

3. Express each of the following in terms of the cosine of another angle between 0° and 180°:

a) –cos 100° b) cos 90° c) –cos 110°
d) – cos 45° e) –cos 122° f) – cos 25°

4. The cosine of which acute angle has the same value as:

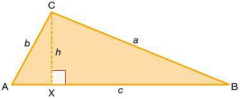
a) cos 125° b) cos 107° c) –cos 120°
d) – cos 98° e) –cos 92° f) – cos 110°?

SECTION 4

The sine and cosine rules

With right-angled triangles we can use the basic trigonometric ratios of sine, cosine and tangent. The **sine rule** is a relationship which can be used with non right-angled triangles.

The sine rule can be derived as follows:



In triangle ACX, $\sin A = \frac{h}{b}$ therefore $h = b \sin A$

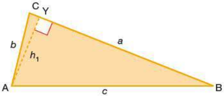
In triangle BCX, $\sin B = \frac{h}{a}$ therefore $h = a \sin B$

As the height h is common to both triangles it can be deduced that:

$b \sin A = a \sin B$

which can be rearranged to: $\frac{a}{\sin A} = \frac{b}{\sin B}$

Similarly when a perpendicular line is drawn from A to meet side BC at Y another equation is formed.



In triangle ACY, $\sin C = \frac{h_1}{b}$ therefore $h_1 = b \sin C$

In triangle BAY, $\sin B = \frac{h_1}{c}$ therefore $h_1 = c \sin B$

As the height h_1 is common to both triangles it can be deduced that:

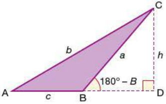
$b \sin C = c \sin B$

which can be rearranged to: $\frac{b}{\sin B} = \frac{c}{\sin C}$

Both equations can be combined to form $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. This is the sine rule.

The reciprocal of each fraction can be taken resulting in in another form of the sine rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

The sine rule proved above also works for obtuse-angled triangles as shown (left).

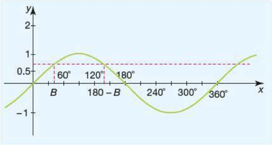


Consider the obtuse-angled triangle ABC.
The height of triangle ABC is h.

From ΔACD $\sin A = \frac{h}{b}$ therefore $h = b\sin A$ (Equation 1)

From ΔBCD $\sin(180 - B) = \frac{h}{a}$ therefore $h = a\sin(180 - B)$ (Equation 2)

However, $\sin(180 - B) = \sin B$. This can be seen from the graph of the sine curve below:

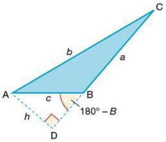


So equation 2 above can be rewritten as $h = a\sin B$ (Equation 3)

Equating equations 1 and 3 gives the equation $b\sin A = a\sin B$

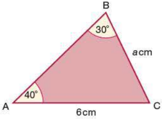
Rearranging this gives the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B}$

By considering the triangle as shown (left) and using a similar proof as above, the sine rule $\frac{b}{\sin B} = \frac{c}{\sin C}$ is derived.



Combining the two results produces $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ as before.

Worked examples



a) Calculate the length of side BC.

Using the sine rule:

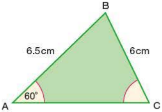
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 40^\circ} = \frac{6}{\sin 30^\circ}$$

$$a = \frac{6 \times \sin 40^\circ}{\sin 30^\circ}$$

$$a = 7.71 \text{ (3 s.f.)}$$

BC = 7.71 cm (3 s.f.)



b) Calculate the size of angle C.

Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Note the reciprocal of both sides has been used.

$$\sin C = \frac{6.5 \times \sin 60^\circ}{6}$$

This makes the subsequent rearrangement easier.

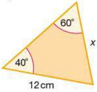
$$C = \sin^{-1}(0.94)$$

$$C = 69.8^\circ \text{ (1 d.p.)}$$

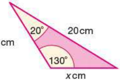
Exercise 8.7

1. Calculate the length of the side marked x in each of the following. Give your answers to 3 s.f.

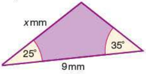
a)



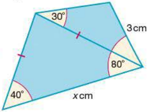
b)



c)

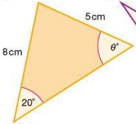


d)

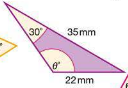


2. Calculate the size of the angle marked $\theta°$ in each of the following. Give your answers to 1 d.p.

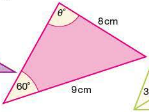
a)



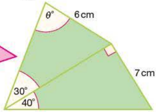
b)



c)



d)



3. Δ ABC has the following dimensions:

AC = 10 cm, AB = 8 cm and ∠ACB = 20°.

a) Calculate the two possible values for ∠CBA.
b) Sketch and label the two possible shapes for Δ ABC.

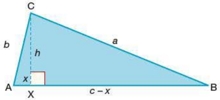
4. Δ PQR has the following dimensions:

PQ = 6 cm, PR = 4 cm and ∠PQR = 40°.

a) Calculate the two possible values for ∠QRP.
b) Sketch and label the two possible shapes for Δ PQR.

The cosine rule

The **cosine rule** is another relationship which can be used with non right-angled triangles.



Using Pythagoras' theorem, two equations can be constructed.

For triangle ACX:

$b^2 = x^2 + h^2$ which rearranged gives: $h^2 = b^2 - x^2$

For triangle BCX:

$a^2 = h^2 + (c - x)^2$ which rearranged gives: $h^2 = a^2 - (c - x)^2$

As h^2 is common to both equations, it can be deduced that:

$$a^2 - (c - x)^2 = b^2 - x^2$$
$$\Rightarrow \quad a^2 = b^2 - x^2 + (c - x)^2$$
$$\Rightarrow \quad a^2 = b^2 - x^2 + (c^2 - 2cx + x^2)$$
$$\Rightarrow \quad a^2 = b^2 + c^2 - 2cx$$

But from triangle ACX, $\cos A = \frac{x}{b}$ therefore $x = b\cos A$.

Substituting $x = b\cos A$ into $a^2 = b^2 + c^2 - 2cx$ gives:

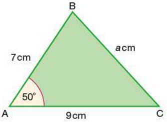
$$a^2 = b^2 + c^2 - 2bc\cos A$$

This is the cosine rule.

If angle A is required the formula can be rearranged to give $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

A similar proof can be applied if angle A is obtuse.

Worked examples **a)** Calculate the length of the side BC.

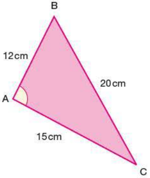


Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 9^2 + 7^2 - (2 \times 9 \times 7 \times \cos 50°)$$
$$= 81 + 49 - (126 \times \cos 50°) = 49.0$$
$$a = \sqrt{49.0}$$
$$a = 7.00 \text{ (3 s.f.)}$$

BC = 7.00 cm (3 s.f.)

b) Calculate the size of angle A.



Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

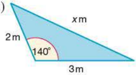
Rearranging the equation gives:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos A = \frac{15^2 + 12^2 - 20^2}{2 \times 15 \times 12} = -0.086$$
$$A = \cos^{-1}(-0.086)$$
$$A = 94.9° \text{ (1 d.p.)}$$

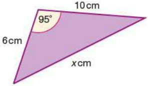
Exercise 8.8

1. Calculate the length of the side marked x in each of the following. Give your answers to 3 s.f.

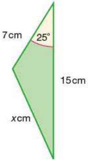
a)



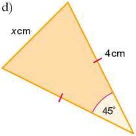
b)



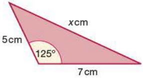
c)



d)

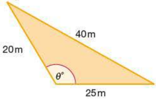


e)

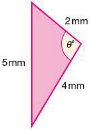


2. Calculate the angle marked $\theta°$ in each of the following. Give your answers to 1 d.p.

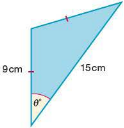
a)



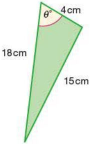
b)



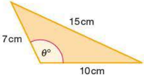
c)



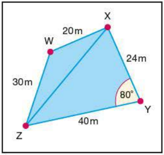
d)



e)

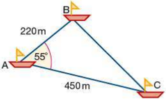


Exercise 8.9



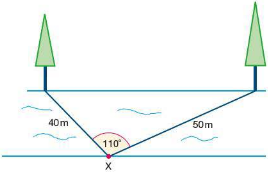
1. Four players, W, X, Y and Z, are on a rugby pitch. The diagram (left) shows a plan view of their relative positions.
Calculate:
a) the distance between players X and Z
b) ∠ZWX
c) ∠WZX
d) ∠YZX
e) the distance between players W and Y.

2. Three yachts, A, B and C, are racing off the 'Cape'. Their relative positions are shown below.



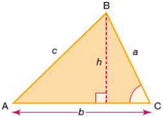
Calculate the distance between B and C to the nearest 10 m.

3. There are two trees standing on one side of a river bank. On the opposite side is a boy standing at X.



Using the information given, calculate the distance between the two trees.

The area of a triangle



The area of a triangle can be calculated without the need for knowing its height.

Area = $\frac{1}{2}bh$

Also: $\sin C = \frac{h}{a}$

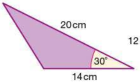
Rearranging gives: $h = a\sin C$

Therefore area = $\frac{1}{2}ab\sin C$

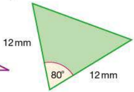
Exercise 8.10

1. Calculate the area of the following triangles. Give your answers to 3 s.f.

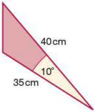
a)



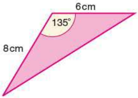
b)



c)

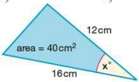


d)

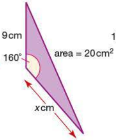


2. Calculate the value of *x* in each of the following. Give your answers correct to 1 d.p.

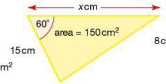
a)



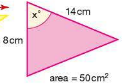
b)

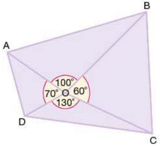


c)



d)





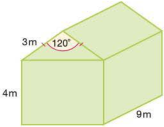
3. ABCD is a school playing field. The following lengths are known:

OA = 83 m, OB = 122 m, OC = 106 m, OD = 78 m

Calculate the area of the school playing field to the nearest 100 m^2.

4. The roof of a garage has a slanting length of 3 m and makes an angle of 120° at its vertex. The height of the garage wall is 4 m and its depth is 9 m.

Calculate:
a) the cross-sectional area of the roof
b) the volume occupied by the whole garage.



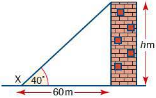
SECTION 5 Applications of trigonometry

Angles of elevation and depression

The **angle of elevation** is the angle above the horizontal through which a line of view is raised. The **angle of depression** is the angle below the horizontal through which a line of view is lowered.

Worked examples

a) The base of a tower is 60 m away from a point X on the ground. If the angle of elevation of the top of the tower from X is 40°, calculate the height of the tower.



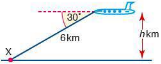
Give your answer to the nearest metre.

$$\tan 40^\circ = \frac{h}{60}$$

$$h = 60 \times \tan 40^\circ = 50$$

The height is 50 m.

b) An aeroplane receives a signal from a point X on the ground. If the angle of depression of point X from the aeroplane is 30°, calculate the height at which the plane is flying.



Give your answer to the nearest 0.1 km.

$$\sin 30^\circ = \frac{h}{6}$$

$$h = 6 \times \sin 30^\circ = 3.0$$

The height is 3.0 km.

Exercise 8.11

1. a) A and B are two villages. If the horizontal distance between them is 12 km, and the vertical distance between them is 2 km calculate:

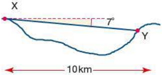
i) the shortest distance between the two villages

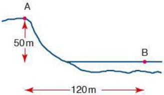
ii) the angle of elevation of B from A.

b) X and Y are two towns. If the horizontal distance between them is 10 km and the angle of depression of Y from X is 7°, calculate:

i) the shortest distance between the two towns

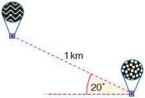
ii) the vertical height between the two towns.

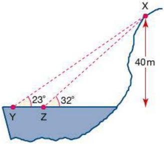




c) A girl standing on a hill at A, overlooking a lake, can see a small boat at a point B on the lake. If the girl is at a height of 50 m above B and at a horizontal distance of 120 m away from B, calculate:
   i) the angle of depression of the boat from the girl
   ii) the shortest distance between the girl and the boat.

d) Two hot air balloons are 1 km apart in the air. If the angle of elevation of the higher from the lower balloon is 20°, calculate the following, giving your answers to the nearest metre:
   i) the vertical height between the two balloons
   ii) the horizontal distance between the two balloons.

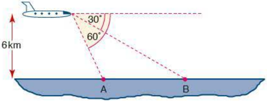




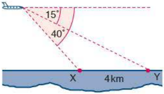
2. a) A boy X can be seen by two of his friends Y and Z, who are swimming in the sea. If the angle of elevation of X from Y is 23° and from Z is 32°, and the height of X above Y and Z is 40 m, calculate:
   i) the horizontal distance between X and Z
   ii) the horizontal distance between Y and Z.

Note: XYZ is a vertical plane.

b) A plane is flying at an altitude of 6 km directly over the line AB. It spots two boats A and B, on the sea.



If the angles of depression of A and B from the plane are 60° and 30° respectively, calculate the horizontal distance between A and B.



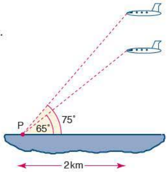
c) A plane is flying at a constant altitude over the sea directly over the line XY. It can see two boats X and Y which are 4 km apart.

If the angles of depression of X and Y from the plane are 40° and 15° respectively, calculate:
   i) the horizontal distance between Y and the plane
   ii) the altitude at which the plane is flying.

d) Two planes are flying directly above each other. A person standing at P can see both of them. The horizontal distance between the two planes and the person is 2 km.

If the angles of elevation of the planes from the person are 65° and 75° calculate:

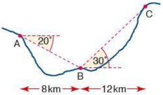
i) the altitude at which the higher plane is flying
ii) the vertical distance between the two planes.



3. a) Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km, and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°.

Calculate, giving all answers to 1 d.p.:

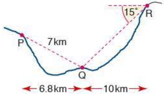
i) the vertical height between A and B
ii) the vertical height between B and C
iii) the angle of elevation of C from A
iv) the shortest distance between A and C.



Note: A, B and C are in the same vertical plane.

b) Using binoculars, three people P, Q and R can see each other across a valley. The horizontal distance between P and Q is 6.8 km and the horizontal distance between Q and R is 10 km. If the shortest distance between P and Q is 7 km and the angle of depression of Q from R is 15°, calculate, giving appropriate answers:

i) the vertical height between Q and R
ii) the vertical height between P and R
iii) the angle of elevation of R from P
iv) the shortest distance between P and R.

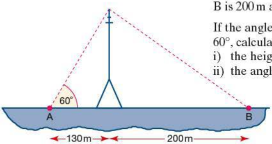


Note: P, Q and R are in the same vertical plane.

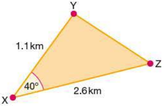
c) Two people A and B are standing either side of a transmission mast. A is 130 m away from the mast and B is 200 m away.

If the angle of elevation of the top of the mast from A is 60°, calculate:

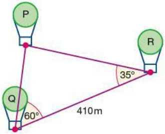
i) the height of the mast to the nearest metre
ii) the angle of elevation of the top of the mast from B.



4. Three boats X, Y and Z are shown below.

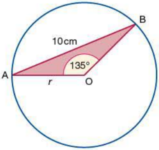


Find the distance between boats Y and Z, giving your answer to the nearest 100 m.

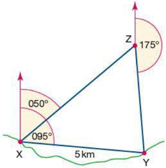


5. Three hot air balloons P, Q and R, travelling in the same vertical plane are shown (left).
If angle PQR = 60°, angle PRQ = 35° and the distance between balloons Q and R is 410 m, calculate:
a) the distance between balloons P and R, to the nearest 10 m
b) the distance between balloons P and Q, to the nearest 10 m.

6. A triangle AOB lies inside a circle. Vertices A and B lie on the circumference of the circle, O at its centre.



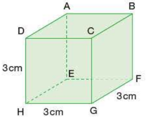
If the angle AOB = 135° and the chord AB = 10 cm, calculate the length of the radius *r*.



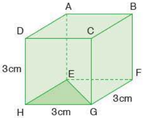
7. The diagram (left) shows two people at X and Y standing 5 km apart on a shore. A large cruise ship is at Z.
The bearing of Y from X is 095°. The bearing of Z from X is 050° and the bearing of Z from Y is 175°.
a) Calculate the angle XZY.
b) Calculate the distance between X and Z. Give your answer to the nearest metre.
c) Calculate the distance between Y and Z. Give your answer to the nearest metre.

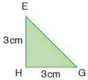
Trigonometry in three dimensions

Worked example The diagram (left) shows a cube of edge length 3 cm.



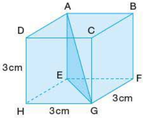
i) Calculate the length EG.
Triangle EHG is right angled. Use Pythagoras' theorem to calculate the length EG:

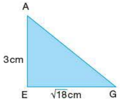




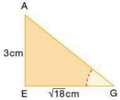
$$EG^2 = EH^2 + HG^2$$
$$EG^2 = 3^2 + 3^2 = 18$$
$$EG = \sqrt{18}\,\text{cm}$$

ii) Calculate the length AG.
Triangle AEG is right angled. Use Pythagoras' theorem to calculate the length AG:





$$AG^2 = AE^2 + EG^2$$
$$AG^2 = 3^2 + (\sqrt{18})^2$$
$$AG^2 = 9 + 18$$
$$AG = \sqrt{27}\,\text{cm}$$



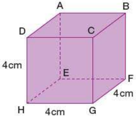
iii) Calculate the angle EGA.
To calculate angle EGA, use the triangle EGA:

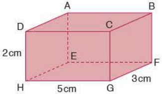
$$\tan G = \frac{3}{\sqrt{18}}$$

$G = 35.3°$ (1 d.p.)

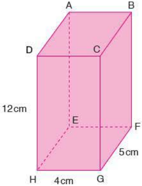
Exercise 8.12

1. a) Calculate the length HF.
   b) Calculate the length of HB.
   c) Calculate the angle BHG.

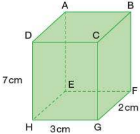




2. a) Calculate the length CA.
   b) Calculate the length CE.
   c) Calculate the angle ACE.

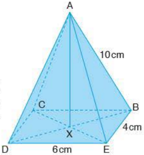


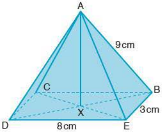
3. a) Calculate the length EG.
   b) Calculate the length AG.
   c) Calculate the angle AGE.



4. a) Calculate the angle BCE.
   b) Calculate the angle GFH.

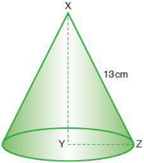
5. The diagram shows a right pyramid where A is vertically above X.
   a) i) Calculate the length DB.
      ii) Calculate the angle DAX.
   b) i) Calculate the angle CED.
      ii) Calculate the angle DBA.



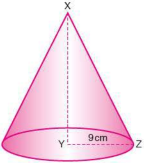


6. The diagram shows a right pyramid where A is vertically above X.
   a) i) Calculate the length CE.
      ii) Calculate the angle CAX.
   b) i) Calculate the angle BDE.
      ii) Calculate the angle ADB.

7. In this cone angle YXZ = 60°.

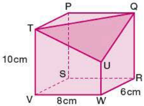


a) Calculate the length XY.
b) Calculate the length YZ.
c) Calculate the circumference of the base.



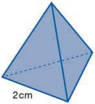
8. In this cone (left) angle XZY = 40°.
a) Calculate the length XZ.
b) Calculate the length XY.

9. One corner of this cuboid has been sliced off along the plane QTU. WU = 4 cm.

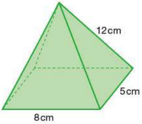


a) Calculate the length of the three sides of the triangle QTU.
b) Calculate the three angles P, Q and T in triangle PQT.
c) Calculate the area of triangle PQT.

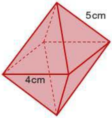
10. Calculate the surface area of a regular tetrahedron with edge length 2 cm.



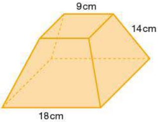
11. The rectangular-based pyramid shown (below) has a sloping edge length of 12 cm. Calculate its surface area.



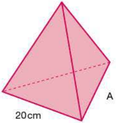
12. Two square-based pyramids are glued together as shown. Given that all the triangular faces are identical, calculate the surface area of the whole shape.

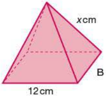


13. Calculate the surface area of the truncated square-based pyramid (below). Assume that all the sloping faces are identical.



14. The two pyramids shown below have the same surface area.

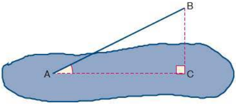




Calculate:

a) the surface area of the tetrahedron
b) the area of one of the triangular faces on the square-based pyramid
c) the value of x.

The angle between a line and a plane

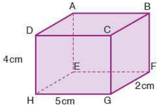


To calculate the size of the angle between the line AB and the shaded plane, drop a perpendicular from B. It meets the shaded plane at C. Then join AC.

The angle between the lines AB and AC represents the angle between the line AB and the shaded plane.

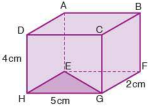
The line AC is the projection of the line AB on the shaded plane.

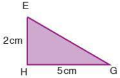
Worked example



i) Calculate the length EC.

First use Pythagoras' theorem to calculate the length EG:

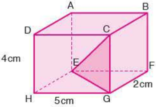


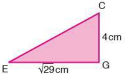


$$EG^2 = EH^2 + HG^2$$
$$EG^2 = 2^2 + 5^2$$
$$EG^2 = 29$$

$$EG = \sqrt{29}\,\text{cm}$$

Now use Pythagoras' theorem to calculate EC:

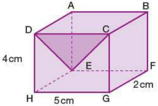


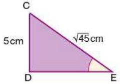


$$EC^2 = EG^2 + CG^2$$
$$EC^2 = (\sqrt{29})^2 + 4^2$$
$$EC^2 = 29 + 16$$
$$EC = \quad 45\,\text{cm}$$

ii) Calculate the angle between the line CE and the plane ADHE.

To calculate the angle between the line CE and the plane ADHE, use the right-angled triangle CED and calculate the angle CED:





$$\sin E = \frac{CD}{CE}$$

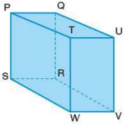
$$\sin E = \frac{5}{\sqrt{45}}$$

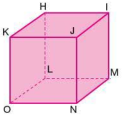
$$E = \sin^{-1}\left(\frac{5}{\sqrt{45}}\right)$$

$$E = 48.2^\circ \text{ (1 d.p.)}$$

Exercise 8.13

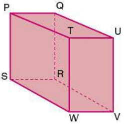
1. Name the projection of each line onto the given plane:
a) TR onto RSWV
b) TR onto PQUT
c) SU onto PQRS
d) SU onto TUVW
e) PV onto QRVU
f) PV onto RSWV



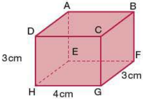


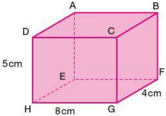
2. Name the projection of each line onto the given plane:
a) KM onto IJNM
b) KM onto JKON
c) KM onto HIML
d) IO onto HLOK
e) IO onto JKON
f) IO onto LMNO

3. Name the angle between the given line and plane:
a) PT and PQRS
b) PU and PQRS
c) SV and PSWT
d) RT and TUVW
e) SU and QRVU
f) PV and PSWT



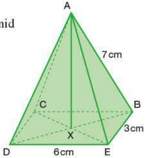
4. a) Calculate the length BH.
b) Calculate the angle between the line BH and the plane EFGH.

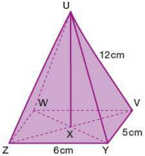




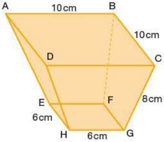
5. a) Calculate the length AG.
b) Calculate the angle between the line AG and the plane EFGH.
c) Calculate the angle between the line AG and the plane ADHE.

6. The diagram shows a right pyramid where A is vertically above X.
a) Calculate the length BD.
b) Calculate the angle between AB and CBED.



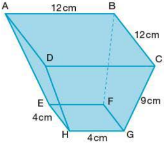


7. The diagram (left) shows a right pyramid where U is vertically above X.
a) Calculate the length WY.
b) Calculate the length UX.
c) Calculate the angle between UX and UZY.



8. ABCD and EFGH are square faces lying parallel to each other.

Calculate:
a) the length DB
b) the length HF
c) the vertical height of the object
d) the angle DH makes with the plane ABCD.

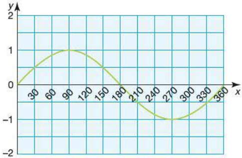


9. ABCD and EFGH are square faces lying parallel to each other.

Calculate:
a) the length AC
b) the length EG
c) the vertical height of the object
d) the angle CG makes with the plane EFGH.

SECTION 6 Trigonometric graphs, properties and transformations

The graphs of the trigonometric ratios $\sin x$, $\cos x$ and $\tan x$ were introduced in Section 3 of this topic. They each have characteristic shapes and properties.



The graph of $y = \sin x$ has:

- a period of 360° (i.e. it repeats itself every 360°)
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\sin x = \sin(180 - x)$.

The graph of $y = \cos x$ has:

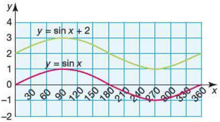
- a period of 360°
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\cos x = \cos(360 - x)$.

The graph of $y = \tan x$ has:

- a period of 180°
- no maximum or minimum value
- symmetry
- asymptotes at 90° and 270°.

In Topic 3 various functions were transformed. These transformations can also be applied to trigonometric graphs.

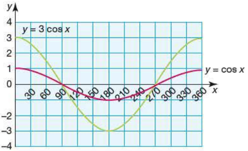
Consider the function $f(x) = \sin x$. The graph of the functions $y = f(x)$ and $y = f(x) + 2$ are shown below.



The graph of $y = \sin x$ has been translated $\begin{pmatrix}0\\2\end{pmatrix}$.

In general the transformation that maps $y = f(x)$ onto $y = f(x) + a$ is the translation $\begin{pmatrix}0\\a\end{pmatrix}$.

Consider the function $f(x) = \cos x$. The graph of the functions $y = f(x)$ and $y = 3f(x)$ are shown below.

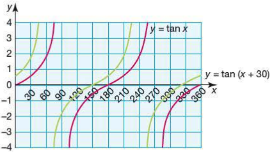


The graph of $y = \cos x$ has been stretched by a scale factor 3, parallel to the y-axis.

In general the transformation that maps $y = f(x)$ onto $y = kf(x)$ is the stretch of scale factor k parallel to the y-axis.

The third type of transformation also involves a translation.

Consider the function $f(x) = \tan x$. The graph of the functions $y = f(x)$ and $y = f(x + 30)$ are shown below. (Note $y = f(x + 30)$ is the same as $y = \tan(x + 30)$.)



The graph of $y = \tan x$ has been translated by $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$.

In general the transformation that maps $y = \mathrm{f}(x)$ onto $y = \mathrm{f}(x - a)$ is the translation $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

Transformations can be investigated more fully using a graphics calculator. Instructions to graph $y = \cos x$ on a graphics calculator are given below:

Casio

MENU 5 to select the graphing mode.

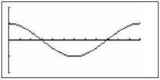
COS X,θ,T EXE to enter the equation $y = \cos x$.

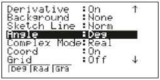
F6 to graph the function.

Note: If the graph does not look the same as the one here, the angle mode may not be set correctly. It can be changed by selecting SHIFT MENU and changing the angle mode to 'Deg'.

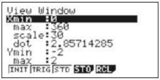
The scale on the axes may also need to be changed by selecting SHIFT F3 and entering the following:

Xmin: 0, Xmax: 360, Xscale: 30, Ymin: –2, Ymax: 2, Yscale: 1.





```
Derivative   :On    ↑
Background   :None
Sketch Line  :Norm
Angle        :Deg
Complex Mode:Real
Coord        :On
Grid         :Off   ↓
Deg Rad Gra
```



```
View Window
Xmin   :0
 max   :360
 scale:30
 dot   :2.85714285
Ymin   :-2
 max   :2
INIT TRIG STD STO RCL
```

Texas

Y= to enter the function COS X,T,θ,n.

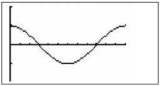
ENTER to enter the equation $y = \cos x$.

GRAPH to graph the function.

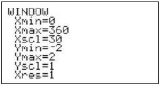
Note: If the graph does not look the same as the one here, the angle mode may not be set correctly. It can be changed by selecting MODE and changing the angle mode to 'Degree'.

The scale on the axes may also need to be changed by selecting WINDOW and entering the following:

Xmin: 0, Xmax: 360, Xscl: 30, Ymin: –2, Ymax: 2, Yscl: 1.



```
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi re^θi
FULL HORIZ G-T
SET CLOCK 23/01/10 21:25
```



```
WINDOW
 Xmin=0
 Xmax=360
 Xscl=30
 Ymin=-2
 Ymax=2
 Yscl=1
 Xres=1
```

Exercise 8.14

1. Determine the transformation which maps $f(x) = \sin x$ onto each of the following functions:

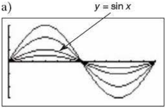
a) $y = 2 \sin x$
b) $y = \sin x - 3$
c) $y = \sin(x + 60)$
d) $y = \cos x$

2. a) Sketch a graph of $f(x) = \cos x$ for values of x between $0 \leqslant x \leqslant 360°$.
b) On the same axes sketch the graph of $y = \frac{1}{2}\cos x$.

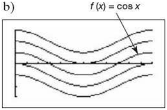
3. a) Sketch a graph of $f(x) = \tan x$ for values of x between $0 \leqslant x \leqslant 360°$.
b) On the same axes sketch the graph of $y = \tan x + 2$.

4. Using your graphics calculator, produce a screen similar to the ones below. One of the functions is identified each time.

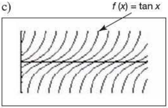
a)



b)



c)



SECTION 7 Investigations, modelling and ICT

Numbered balls

The balls above start with the number 25 and then subsequent numbered balls are added according to a rule. The process stops when ball number 1 is added.

1. Express in words the rule for generating the sequence of numbered balls.
2. What is the longest sequence of balls starting with a number less than 100?

3. Is there a strategy for generating a long sequence?
4. Use your rule to state the longest sequence of balls starting with a number less than 1000.
5. Extend the investigation by having a different term-to-term rule.

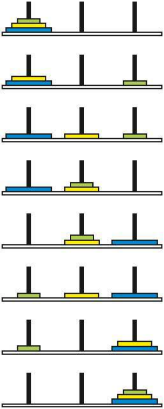
Towers of Hanoi

This investigation is based on an old Vietnamese legend. The legend is as follows:

> At the beginning of time a temple was created by the Gods. Inside the temple stood three giant rods. On one of these rods, 64 gold discs, all of different diameters, were stacked in descending order of size, i.e the largest at the bottom rising to the smallest at the top. Priests at the temple were responsible for moving the discs onto the remaining two rods until all 64 discs were stacked in the same order on one of the other rods. When this task was completed, time would cease and the world would come to an end.

The discs however could only be moved according to certain rules. These were:

- Only one disc could be moved at a time.
- A disc could only be placed on top of a larger one.



The diagram (left) shows the smallest number of moves required to transfer three discs from the rod on the left to the rod on the right.

With three discs, the smallest number of moves is seven.

1. What is the smallest number of moves needed for 2 discs?
2. What is the smallest number of moves needed for 4 discs?
3. Investigate the smallest number of moves needed to move different numbers of discs.
4. Display the results of your investigation in an ordered table.
5. Describe any patterns you see in your results.
6. Predict, from your results, the smallest number of moves needed to move 10 discs.
7. Determine a formula for the smallest number of moves for *n* discs.
8. Assume the priests have been transferring the discs at the rate of one per second and assume the Earth is approximately 4.54 billion years old (4.54×10^9 years).

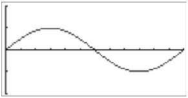
According to the legend, is the world coming to an end soon? Justify your answer with relevant calculations.

ICT Activity

In this activity you will need to use your graphics calculator to investigate the relationship between different trigonometric ratios.

1. a) Using your calculator, plot the graph of $y = \sin x$ for $0° \leqslant x \leqslant 360°$.

The graph should look similar to the one shown below:



b) Using the graph solving facility evaluate the following:

i) sin 70°

ii) sin 125°

iii) sin 300°

c) Referring to the graph explain why $\sin x = 0.7$ has two solutions between 0° and 360°.

d) Use the graph to solve the equation $\sin x = 0.5$.

2. a) On the same axes as before plot $y = \cos x$.

b) How many solutions are there to the equation $\sin x = \cos x$ between 0° and 360°?

c) What is the solution to the equation $\sin x = \cos x$ between 180° and 270°?

3. By plotting appropriate graphs solve the following for $0° \leqslant x \leqslant 360°$.

a) $\sin x = \tan x$

b) $\cos x = \tan x$

SECTION
8

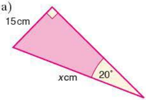
Student assessments

Student assessment 1

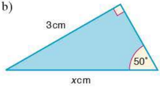
NB: Diagrams are not drawn to scale.

1. Calculate the length of the side marked x cm in these diagrams. Give your answers correct to 3 s.f.

a)

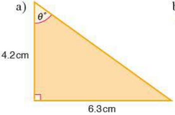


b)

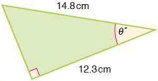


2. Calculate the angle marked $\theta°$ in these diagrams. Give your answers correct to the nearest degree.

a)

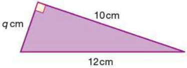


b)

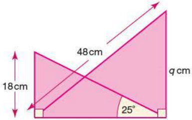


3. Calculate the length of the side marked q cm in these diagrams. Give your answers correct to 3 s.f.

a)



b)



4. Two trees stand 16 m apart. Their tops make an angle of $\theta°$ at point A on the ground.

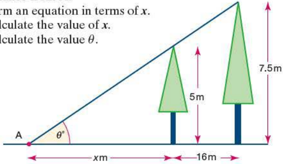
a) Express $\theta°$ in terms of the height of the shorter tree and its distance x metres from point A.

b) Express $\theta°$ in terms of the height of the taller tree and its distance from A.

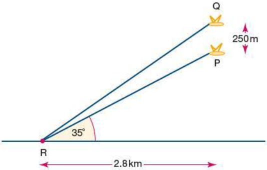
c) Form an equation in terms of x.

d) Calculate the value of x.

e) Calculate the value θ.



5. Two hawks P and Q are flying vertically above one another. Hawk Q is 250 m above hawk P. They both spot a snake at R.

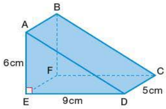


Using the information given, calculate:

a) the height of P above the ground

b) the distance between P and R

c) the distance between Q and R.



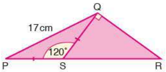
Student assessment 2

1. Using the triangular prism (left) calculate:
   a) the length AD
   b) the length AC
   c) the angle AC makes with the plane CDEF
   d) the angle AC makes with the plane ABFE.

2. Draw a graph of $y = \cos \theta°$, for $0° \leqslant \theta° \leqslant 360°$. Mark on the angles 0°, 90°, 180°, 270°, 360° and also the maximum and minimum values of y.

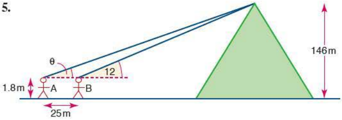
3. The cosine of which other angle between 0 and 180° has the same value as:
   a) cos 128° b) –cos 80°?

4. For the triangle below calculate:



   a) the length PS
   b) ∠QRS
   c) the length SR.

5.

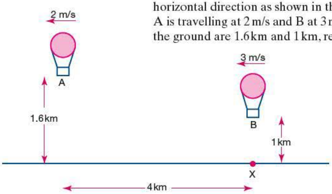


The Great Pyramid at Giza is 146m high. Two people A and B are looking at the top of the pyramid. The angle of elevation of the top of the pyramid from B is 12°.
The distance between A and B is 25m. If both A and B are 1.8m tall, calculate:
   a) the distance from B to the centre of the base of the pyramid
   b) the angle of elevation θ of the top of the pyramid from A
   c) the distance between A's head and the top of the pyramid.

Note: A, B and the top of the pyramid are in the same vertical plane.

Student assessment 3

1. Two hot air balloons A and B are travelling in the same horizontal direction as shown in the diagram below. A is travelling at 2 m/s and B at 3 m/s. Their heights above the ground are 1.6 km and 1 km, respectively.

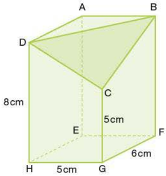


At midday, their horizontal distance apart is 4 km and balloon B is directly above a point X on the ground.

Calculate:

a) the angle of elevation of A from X at midday
b) the angle of depression of B from A at midday
c) their horizontal distance apart at 12.30 p.m.
d) the angle of elevation of B from X at 12.30 p.m.
e) the angle of elevation of A from B at 12.30 p.m.
f) how much closer A and B are at 12.30 p.m. compared with midday.

2. a) Plot the graphs of $y = \sin\theta°$ and $y = \cos\theta°$, for $0° \leqslant \theta° \leqslant 360°$, on the same axes.
   b) Use your graph to find the angles for which $\sin\theta° = \cos\theta°$.



3. This cuboid has one of its corners removed to leave a flat triangle BDC.

Calculate:

a) the length DC
b) the length BC
c) the length DB
d) $\angle$CBD
e) the area of Δ BDC
f) the angle AC makes with the plane AEHD.

4. Describe the transformation that maps $f(x) = \cos x$ onto each of the following graphs:
   a) $y = \cos x - 5$
   b) $y = \frac{1}{4}\cos x$
   c) $y = \cos(x + 120)$
   d) $y = \sin x$

TOPIC

9 Sets

This topic will cover the following syllabus content:

9.1 Notation and meaning for:
  is an element of ($\in$); is not an element of ($\notin$);
  is a subset of ($\subseteq$); is a proper subset of ($\subset$);
  universal set U, empty set $\varnothing$ or { };
  complement of A, (A'); number of elements in A, $n(A)$
9.2 Sets in descriptive form { x | } or as a list
9.3 Venn diagrams with at most three sets
9.4 Intersection and union of sets

Sections

SECTION 1 The Germans

Carl Gauss (1777–1855)

Carl Gauss (1777–1855)

Gauss was considered to be a genius equal to Isaac Newton. He discovered the Fundamental Theorem of Algebra and also worked in statistics and differential geometry. He also solved astronomical problems related to comet orbits and navigation by the stars.

He was not interested in fame so did not bother to publish much of his work. He wrote 'It is not knowledge, but the act of learning which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it.' He did not care that other people made the same discoveries later and took all the credit.

Gauss discovered very advanced mathematics including, non-Euclidean geometry, a prime number formula, the foundations of topology, and the first ideas of knot theory.

Bernhard Riemann (1826–1866)

Bernhard Riemann was one of the most talented mathematicians. However, he had poor health and died at the age of forty. He applied topology to analysis and then applied analysis to number theory. He also worked on differential geometry, non-Euclidean geometry and the theory of functions.

Riemann was also interested in Physics and described a new geometry of space.

He proposed a theory unifying electricity, magnetism and light. Modern physics, beginning with Einstein's relativity, relies on Riemann's ideas of the geometry of space.

Georg Cantor (1845–1918)

Georg Cantor is of major importance because he created modern set theory which is studied in this topic. He was the first to prove that there must be more real numbers than integers.

SECTION 2

Sets, subsets and Venn diagrams

A **set** is a well defined group of objects or symbols. The objects or symbols are called the **elements** of the set. If an element e belongs to a set S, this is represented as $e \in S$. If e does not belong to set S this is represented as $e \notin S$.

Worked examples

a) A particular set consists of the following elements:
{South Africa, Namibia, Egypt, Angola, …}

i) Describe the set.
The elements of the set are countries of Africa.

ii) Add another two elements to the set.
e.g. Zimbabwe, Ghana

iii) Is the set finite or infinite?
Finite. There is a finite number of countries in Africa.

b) Consider the set {1, 4, 9, 16, 25, …}.

i) Describe the set.
The elements of the set are square numbers.

ii) Write another two elements of the set.
e.g. 36, 49

iii) Is the set finite or infinite?
Infinite. There is an infinite number of square numbers.

Exercise 9.1

1. In the following questions:
i) describe the set in words
ii) write down another two elements of the set.

a) {Asia, Africa, Europe, …}
b) {2, 4, 6, 8, …}
c) {Sunday, Monday, Tuesday, …}
d) {January, March, July, …}
e) {1, 3, 6, 10, …}
f) {Mehmet, Michael, Mustapha, Matthew, …}
g) {11, 13, 17, 19, …}
h) {a, e, i, …}
i) {Earth, Mars, Venus, …}
j) $A = \{x \mid 3 \leqslant x \leqslant 12\}$
k) $S = \{y \mid -5 \leqslant y \leqslant 5\}$

2. The number of elements in a set A is written as $n(A)$.
Give the value of $n(A)$ for the finite sets in Q.1(a)–(k).

Subsets

If all the elements of one set X are also elements of another set Y, then X is said to be a **subset** of Y.

This is written as $X \subseteq Y$.

If a set A is empty (i.e. it has no elements in it), then this is called the empty set and it is represented by the symbol $\varnothing$. Therefore $A = \varnothing$. The empty set is a subset of all sets.

e.g. Three girls, Winnie, Natalie and Emma form a set A:
$A = \{\text{Winnie, Natalie, Emma}\}$
All the possible subsets of A are given below:
$B = \{\text{Winnie, Natalie, Emma}\}$
$C = \{\text{Winnie, Natalie}\}$
$D = \{\text{Winnie, Emma}\}$
$E = \{\text{Natalie, Emma}\}$
$F = \{\text{Winnie}\}$
$G = \{\text{Natalie}\}$
$H = \{\text{Emma}\}$
$I = \varnothing$

Note that the sets B and I above are considered as subsets of A,

i.e. $B \subseteq A$ and $I \subseteq A$.

However, sets C, D, E, F, G, H and I are considered **proper subsets** of A.

A proper subset is a subset which is not the same as the original set itself.

This distinction of subset is shown in the notation below:

$C \subset A$ and $I \subset A$, etc.

Similarly $G \nsubseteq H$ implies that G is not a subset of H
$G \not\subset H$ implies that G is not a proper subset of H

Worked example $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

i) List subset B {even numbers}.
$B = \{2, 4, 6, 8, 10\}$

ii) List subset C {prime numbers}.
$C = \{2, 3, 5, 7\}$

Exercise 9.2

1. P = {whole numbers less than 30}
a) List the subset Q {even numbers}.
b) List the subset R {odd numbers}.
c) List the subset S {prime numbers}.
d) List the subset T {square numbers}.
e) List the subset U {triangle numbers}.

2. $A = \{\text{whole numbers between 50 and 70}\}$
   a) List the subset B {multiples of 5}.
   b) List the subset C {multiples of 3}.
   c) List the subset D {square numbers}.

3. $J = \{\mathrm{p}, \mathrm{q}, \mathrm{r}\}$
   a) List all the subsets of J.
   b) List all the proper subsets of J.

4. State whether each of the following statements is true or false:
   a) $\{\text{Algeria, Mozambique}\} \subseteq \{\text{countries in Africa}\}$
   b) $\{\text{mango, banana}\} \subseteq \{\text{fruit}\}$
   c) $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$
   d) $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$
   e) $\{\text{volleyball, basketball}\} \nsubseteq \{\text{team sport}\}$
   f) $\{4, 6, 8, 10\} \not\subset \{4, 6, 8, 10\}$
   g) $\{\text{potatoes, carrots}\} \subseteq \{\text{vegetables}\}$
   h) $\{12, 13, 14, 15\} \not\subset \{\text{whole numbers}\}$

The universal set

The **universal set** (U) for any particular problem is the set which contains all the possible elements for that problem.

The **complement** of a set A is the set of elements which are in U but not in A. The set is identified as A'. Notice that $U' = \varnothing$ and $\varnothing' = U$.

Worked examples

a) If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4, 5\}$, what set is represented by A'?

A' consists of those elements in U which are not in A.
Therefore $A' = \{6, 7, 8, 9, 10\}$.

b) If $U = \{\text{all 3D shapes}\}$ and $P = \{\text{prisms}\}$, what set is represented by P'?

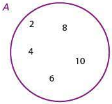
$P' = \{\text{all 3D shapes except prisms}\}$.

Venn diagrams

Venn diagrams are the principal way of showing sets diagrammatically. The method consists primarily of entering the elements of a set into a circle or circles.

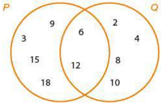
Some examples of the uses of Venn diagrams are shown on the next page.

$A = \{2, 4, 6, 8, 10\}$ can be represented as:



Elements which are in more than one set can also be represented using a Venn diagram.

$P = \{3, 6, 9, 12, 15, 18\}$ and $Q = \{2, 4, 6, 8, 10, 12\}$ can be represented as:

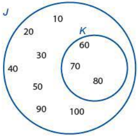


In the diagram above it can be seen that those elements which belong to both sets are placed in the region of overlap of the two circles.

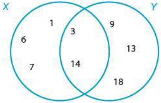
When two sets P and Q overlap as they do above, the notation $P \cap Q$ is used to denote the set of elements in the intersection, i.e. $P \cap Q = \{6, 12\}$.

Note that $6 \in P \cap Q; 8 \notin P \cap Q$.

$J = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$ and
$K = \{60, 70, 80\}$; as discussed earlier, $K \subset J$ can be represented as:



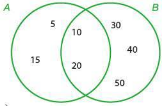
$X = \{1, 3, 6, 7, 14\}$ and $Y = \{3, 9, 13, 14, 18\}$ are represented as:



The **union** of two sets is everything which belongs to either or both sets and is represented by the symbol $\cup$.

Therefore in the example at the bottom of page 395, $X \cup Y = \{1, 3, 6, 7, 9, 13, 14, 18\}$.

Exercise 9.3



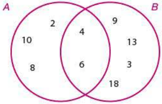
1. Using the Venn diagram (left), indicate whether the following statements are true or false. ($\in$ means 'is an element of' and $\notin$ means 'is not an element of'.)

a) $5 \in A$ b) $20 \in B$ c) $20 \notin A$

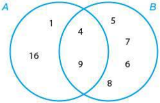
d) $50 \in A$ e) $50 \notin B$ f) $A \cap B = \{10, 20\}$

2. Complete the statement $A \cap B = \{...\}$ for each of the Venn diagrams below:

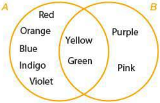
a)

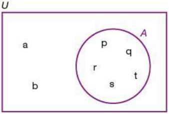


b)



c)





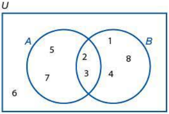
3. Copy and complete the statement $A \cup B = \{...\}$ for each of the Venn diagrams in Q.2 above.

4. Using the Venn diagram (left), copy and complete the following statements:

a) $U = \{...\}$

b) $A' = \{...\}$

5. Using this Venn diagram, copy and complete the following statements:



a) $U = \{...\}$

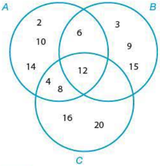
b) $A' = \{...\}$

c) $A \cap B = \{...\}$

d) $A \cup B = \{...\}$

e) $(A \cap B)' = \{...\}$

f) $A \cap B' = \{...\}$



6. a) Describe in words the elements of:

i) set A ii) set B iii) set C

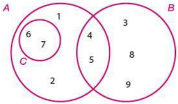
b) Copy and complete the following statements:

i) $A \cap B = \{...\}$ ii) $A \cap C = \{...\}$

iii) $B \cap C = \{...\}$ iv) $A \cap B \cap C = \{...\}$

v) $A \cup B = \{...\}$ vi) $C \cup B = \{...\}$

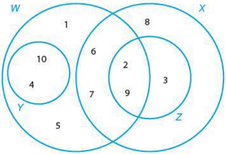
7.



a) Copy and complete the following statements:
   i) $A = \{...\}$ ii) $B = \{...\}$ iii) $C' = \{...\}$
   iv) $A \cap B = \{...\}$ v) $A \cup B = \{...\}$ vi) $(A \cap B)' = \{...\}$

b) State, using set notation, the relationship between C and A.

8.



a) Copy and complete the following statements:
   i) $W = \{...\}$ ii) $X = \{...\}$ iii) $Z' = \{...\}$
   iv) $W \cap Z = \{...\}$ v) $W \cap X = \{...\}$ vi) $Y \cap Z = \{...\}$

b) Which of the named sets is a subset of X?

Exercise 9.4

1. A = {Egypt, Libya, Morocco, Chad}
   B = {Iran, Iraq, Turkey, Egypt}
   a) Draw a Venn diagram to illustrate the above information.
   b) Copy and complete the following statements:
      i) $A \cap B = \{...\}$ ii) $A \cup B = \{...\}$

2. $P = \{2, 3, 5, 7, 11, 13, 17\}$
   $Q = \{11, 13, 15, 17, 19\}$
   a) Draw a Venn diagram to illustrate the above information.
   b) Copy and complete the following statements:
      i) $P \cap Q = \{...\}$ ii) $P \cup Q = \{...\}$

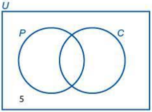
3. $B = \{2, 4, 6, 8, 10\}$
   $A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$
   $A \cap B = \{2, 4\}$
   Represent the above information on a Venn diagram.

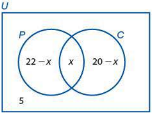
4. $X = \{a, c, d, e, f, g, l\}$
   $Y = \{b, c, d, e, h, i, k, l, m\}$
   $Z = \{c, f, i, j, m\}$
   Represent the above information on a Venn diagram.

5. $P = \{1, 4, 7, 9, 11, 15\}$
   $Q = \{5, 10, 15\}$
   $R = \{1, 4, 9\}$
   Represent the above information on a Venn diagram.

Problems involving sets

Worked example





In a class of 31 students, some study Physics and some study Chemistry. If 22 study Physics, 20 study Chemistry and 5 study neither, calculate the number of students who take both subjects.

The information given above can be entered in a Venn diagram in stages.

The students taking neither Physics nor Chemistry can be put in first.

This leaves 26 students to be entered into the set circles. If x students take both subjects then:

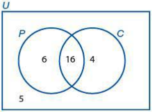
$$n(P) = 22 - x + x$$
$$n(C) = 20 - x + x$$
$$P \cup C = 31 - 5 = 26$$

Note $n(P)$ means the number of elements in set P.

Therefore $22 - x + x + 20 - x = 26$

$$42 - x = 26$$
$$x = 16$$

Substituting the value of x into the Venn diagram gives:



Therefore the number of students taking both Physics and Chemistry is 16.

Exercise 9.5

1. In a class of 35 students, 19 take Spanish, 18 take French and 3 take neither. Calculate how many take:
   a) both French and Spanish
   b) just Spanish
   c) just French.

2. In a year group of 108 students, 60 liked football, 53 liked running and 10 liked neither. Calculate the number of students who liked football but not running.

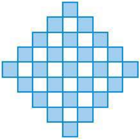
3. In a year group of 113 students, 60 liked badminton, 45 liked rugby and 18 liked neither. Calculate the number of students who:
   a) liked both badminton and rugby
   b) liked only badminton.

4. One year, 37 students sat an examination in Physics, 48 sat an examination in Chemistry and 45 sat an examination in Biology. 15 students sat examinations in Physics and Chemistry, 13 sat examinations in Chemistry and Biology, 7 sat examinations in Physics and Biology and 5 students sat examinations in all three.
   a) Draw a Venn diagram to represent this information.
   b) Calculate $n(P \cup C \cup B)$.

SECTION 3 Investigations, modelling and ICT

Coloured tiles

A large floor space is covered in two different coloured square tiles in a similar way to that shown below:



This pattern consists of blue and white tiles and is 9 tiles across.

1. a) How many blue tiles are there?
   b) How many white tiles are there?

The actual floor is 99 tiles across.

2. By investigating floors of different sizes, determine:
   a) the number of blue tiles on the actual floor
   b) the number of white tiles on the actual floor.

3. For a floor that is *n* tiles across, determine:
  a) the number of blue tiles in terms of *n*
  b) the number of white tiles in terms of *n*.

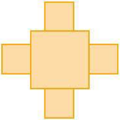
4. Prove, using diagrams, why the rules obtained in Q.3 work.

Fractal patterns

Consider a square of side length 1 cm.

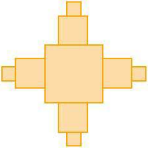
The perimeter of the shape is 4 cm

A square of side length $\frac{1}{2}$cm is added to the centre of each of the existing sides as shown.

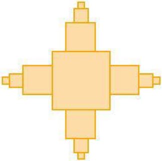


1. Calculate the perimeter of the new shape.

A square of side length $\frac{1}{4}$cm is added to the centre of the outside edge of each of the new squares as shown:



2. Calculate the perimeter of this shape.
   The pattern is continued. Each time the side length of each square is halved and added on to the outside edge. The next stage is shown below:



3. What is the perimeter of the new shape?
4. Enter your results in a table similar to the one shown below:

Pattern	Size of square added	Perimeter
1	1 × 1	4
2	$\frac{1}{2} \times \frac{1}{2}$	
3	$\frac{1}{4} \times \frac{1}{4}$	
4	$\frac{1}{8} \times \frac{1}{8}$	

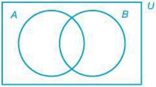
5. Investigate the perimeter of the patterns at different stages.
6. Describe any patterns you see as a result of your investigation
7. Predict, without drawing, the perimeter of the 10th pattern.
8. Determine the rule for the perimeter of the *nth* pattern
9. What is the maximum perimeter these patterns can have? Explain your answer.

SECTION 4 Student assessments

Student assessment 1

1. Describe the following sets in words:
   a) $\{1, 3, 5, 7\}$
   b) $\{1, 3, 5, 7, ...\}$
   c) $\{1, 3, 6, 10, 15, ...\}$
   d) {Brazil, Chile, Argentina, Bolivia,...}

2. Calculate the value of $n(A)$ for each of the sets shown below:
   a) A = {months of the year}
   b) A = {square numbers between 99 and 149}
   c) $A = \{x \mid x \text{ is an integer and } -9 \leqslant x \leqslant -3\}$
   d) A = {students in your class}



3. Copy the Venn diagram (left) twice.
   a) On one copy, shade and label the region which represents U.
   b) On the other copy, shade and label the region which represents $(A \cap B)'$.

4. If A = {w, o, r, k} list all the subsets of A with at least three elements.

5. If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $P = \{2, 4, 6, 8\}$, what set is represented by P'?

Student assessment 2

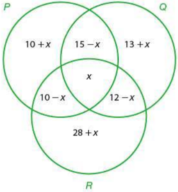
1. M = {a, e, i, o, u}
   a) How many subsets are there of M?
   b) List the subsets of M with four or more elements.

2. X = {lion, tiger, cheetah, leopard, puma, jaguar, cat}
   Y = {elephant, lion, zebra, cheetah, gazelle}
   Z = {anaconda, jaguar, tarantula, mosquito}
   a) Draw a Venn diagram to represent the above information.
   b) Copy and complete the statement $X \cap Y = \{...\}$.
   c) Copy and complete the statement $Y \cap Z = \{...\}$.
   d) Copy and complete the statement $X \cap Y \cap Z = \{...\}$.

3. A group of 40 people were asked whether they like tennis (T) and football (F). The number liking both tennis and football was three times the number liking only tennis. Adding 3 to the number liking only tennis and doubling the answer equals the number of people liking only football. Four said they did not like sport at all.

a) Draw a Venn diagram to represent this information.
b) Calculate $n(T \cap F)$.
c) Calculate $n(T \cap F')$.
d) Calculate $n(T' \cap F)$.

4. The Venn diagram below shows the number of elements in three sets P, Q and R.



If $n(P \cup Q \cup R) = 93$ calculate:

a) x
b) $n(P)$
c) $n(Q)$
d) $n(R)$
e) $n(P \cap Q)$
f) $n(Q \cap R)$
g) $n(P \cap R)$
h) $n(R \cup Q)$
i) $n(P \cap Q)'$

TOPIC

10 Probability

This topic will cover the following syllabus content:

10.1 Probability P(A) as a fraction, decimal or percentage
Significance of its value

10.2 Relative frequency as an estimate of probability

10.3 Expected number of occurrences

10.4 Combining events:
the addition rule $P(A \text{ or } B) = P(A) + P(B)$
the multiplication rule $P(A \text{ and } B) = P(A) \times P(B)$

10.5 Tree diagrams including successive selection with or without replacement

10.6 Probabilities from Venn diagrams and tables

Sections

SECTION 1

Order and chaos

Blaise Pascal (1623–1662)

Blaise Pascal and Pierre de Fermat (known for his last theorem) corresponded about problems connected to games of chance.

Although Newton and Galileo had had some thoughts on the subject, this is accepted as the beginning of the study of what is now called probability. Later, in 1657, Christiaan Huygens wrote the first book on the subject entitled *The Value of all Chances in Games of Fortune*.

In 1821 Carl Friedrich Gauss (1777–1855) worked on normal distribution.

At the start of the nineteenth century, the French mathematician Pierre Simon de Laplace was convinced of the existence of a Newtonian universe. In other words, if you knew the position and velocities of all the particles in the universe, you would be able to predict the future because their movement would be predetermined by scientific laws. However, quantum mechanics has since shown that this is not true. Chaos theory is at the centre of understanding these limits.

SECTION 2

Theoretical probability

Probability of an event

Probability is the study of chance, or the likelihood of an event happening.

In this section we will be looking at theoretical probability. But, because probability is based on chance, what theory predicts does not necessarily happen in practice.

A favourable outcome refers to the event in question actually happening. The total number of possible outcomes refers to all the different types of outcome one can get in a particular situation. In general:

$$\text{Probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of equally likely outcomes}}$$

Therefore

if the probability = 0, it implies the event is impossible
if the probability = 1, it implies the event is certain to happen.

Worked examples

a) An ordinary, fair dice is rolled.

i) Calculate the probability of getting a 6.

Number of favourable outcomes = 1 (i.e. getting a 6)

Total number of possible outcomes = 6
(i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of getting a 6, $P(6) = \frac{1}{6}$

ii) Calculate the probability of not getting a 6.

Number of favourable outcomes = 5
(i.e. getting a 1, 2, 3, 4, 5)

Total number of possible outcomes = 6
(i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of not getting a six, $P(6') = \frac{5}{6}$

Note: the notation P(6')

From this it can be seen that the probability of not getting a 6 is equal to 1 minus the probability of getting a 6.

i.e. $P(6) = 1 - P(6')$

These are known as **complementary events**.
In general, for an event A, $P(A) = 1 - P(A')$

Exercise 10.1

1. Calculate the theoretical probability, when rolling an ordinary, fair dice, of getting each of the following:
- a) a score of 1
- b) a score of 5
- c) an odd number
- d) a score less than 6
- e) a score of 7
- f) a score less than 7.

2. a) Calculate the probability of:
  - i) being born on a Wednesday
  - ii) not being born on a Wednesday.

b) Explain the result of adding the answers to Q.2 a (i) and (ii) together.

3. 250 tickets are sold for a raffle. What is the probability of winning if you buy:
- a) 1 ticket
- b) 5 tickets
- c) 250 tickets
- d) 0 tickets?

4. In a class there are 25 girls and 15 boys. The teacher collects all of their workbooks in a random order. Calculate the probability that the teacher will:
- a) mark a book belonging to a girl first
- b) mark a book belonging to a boy first.

5. Tiles, each lettered with one different letter of the alphabet, are put into a bag. If one tile is drawn out at random, calculate the probability that it is:
a) an A or P
b) a vowel
c) a consonant
d) an X, Y or Z
e) a letter in your first name.

6. A boy was late for school five times in the previous 30 school days. If tomorrow is a school day, calculate the probability that he will arrive late.

7. a) 3 red, 10 white, 5 blue and 2 green counters are put into a bag. If one is picked at random, calculate the probability that it is:
  i) a green counter
  ii) a blue counter.
b) If the first counter taken out is green and it is not put back into the bag, calculate the probability that the second counter picked is:
  i) a green counter
  ii) a red counter.

8. A spinner has the numbers 0 to 36 equally spaced around its edge. Assuming that it is unbiased, calculate the probability on spinning it of it stopping on:
a) the number 5
b) an even number
c) an odd number
d) zero
e) a number greater than 15
f) a multiple of 3
g) a multiple of 3 or 5
h) a prime number.

9. The letters R, C and A can be combined in several different ways.
a) Write the letters in as many different combinations as possible.
b) If a computer writes these three letters at random, calculate the probability that:
  i) the letters will be written in alphabetical order
  ii) that the letter R is written before both the letters A and C
  iii) that the letter C is written after the letter A
  iv) the computer will spell the word CART if the letter T is added.

10. A normal pack of playing cards contains 52 cards. These are made up of four suits (hearts, diamonds, clubs and spades). Each suit consists of 13 cards. These are labelled ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The hearts and diamonds are red; the clubs and spades are black.

If a card is picked at random from a normal pack of cards, calculate the probability of picking:

a) a heart
b) a black card
c) a four
d) a red King
e) a Jack, Queen or King
f) the ace of spades
g) an even numbered card
h) a seven or a club.

Combined events

Here we look at the probability of two or more events happening: combined events. If only two events are involved, then two-way tables can be used to show the outcomes.

Worked example

i) Two coins are tossed. Show all the possible outcomes in a two-way table.

		Coin 1	
		Head	Tail
Coin 2	Head	HH	TH
	Tail	HT	TT

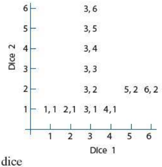
ii) Calculate the probability of getting two heads, P(HH).
All four outcomes are equally likely, therefore P(HH) = $\frac{1}{4}$.

iii) Calculate the probability of getting a head and a tail in any order.
The probability of getting a head and a tail in any order, i.e. HT or TH, is $\frac{2}{4} = \frac{1}{2}$.

Exercise 10.2

1. a) Two fair tetrahedral dice are rolled. If each is numbered 1–4, draw a two-way table to show all the possible outcomes.
b) What is the probability that both dice show the same number?
c) What is the probability that the number on one dice is double the number on the other?
d) What is the probability that the sum of both numbers is prime?

2. Two fair dice are rolled. Copy and complete the diagram (right) to show all the possible combinations.

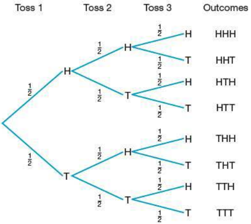


What is the probability of getting:
a) a double 3
b) any double
c) a total score of 11
d) a total score of 7
e) an even number on both dice
f) an even number on at least one dice
g) a 6 or a double
h) scores which differ by 3
i) a total which is either a multiple of 2 or 5?

SECTION 3 Tree diagrams

When more than two combined events are being considered, two-way tables cannot be used and therefore another method of representing information diagrammatically is needed. Tree diagrams are a good way of doing this.

Worked example **i)** If a coin is tossed three times, show all the possible outcomes on a tree diagram, writing each of the probabilities at the side of the branches.



ii) What is the probability of getting three heads?

To calculate the probability of getting three heads multiply along the branches:

$P(HHH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

iii) What is the probability of getting two heads and one tail in any order?

The successful outcomes are HHT, HTH, THH.

$$P(HHT) + P(HTH) + P(THH) = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{8}$$

Therefore the probability is $\frac{3}{8}$.

iv) What is the probability of getting at least one head?

This refers to any outcome with either one, two or three heads, i.e. all of them except TTT.

P (at least one head) $= 1 - P(TTT) = 1 - \frac{1}{8} = \frac{7}{8}$

Therefore the probability is $\frac{7}{8}$.

v) What is the probability of getting no heads?

The only successful outcome for this event is TTT.

Therefore the probability is $\frac{1}{8}$.

Exercise 10.3

1. a) A computer uses the numbers 1, 2 or 3 at random to make three-digit numbers. Assuming that a number can be repeated, show on a tree diagram all the possible combinations that the computer can print.

b) Calculate the probability of getting:

i) the number 131
ii) an even number
iii) a multiple of 11
iv) a multiple of 3
v) a multiple of 2 or 3
vi) a palindromic number.

2. a) A family has four children. Draw a tree diagram to show all the possible combinations of boys and girls. [Assume P (girl) = P (boy).]

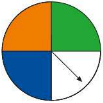
b) Calculate the probability of getting:

i) all girls
ii) two girls and two boys
iii at least one girl
iv) more girls than boys.

3. a) A netball team plays three matches. In each match the team is equally likely to win, lose or draw. Draw a tree diagram to show all the possible outcomes over the three matches.

b) Calculate the probability that the team:
  - i) wins all three matches
  - ii) wins more times than it loses
  - iii) loses at least one match
  - iv) either draws or loses all the three matches.

c) Explain why it is not very realistic to assume that the outcomes are equally likely in this case.



4. A spinner is split into quarters.

a) If it is spun twice, draw a probability tree showing all the possible outcomes.

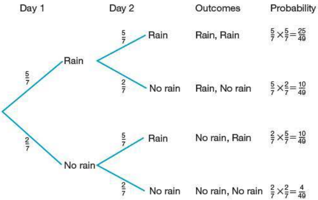
b) Calculate the probability of getting:
  - i) two greens
  - ii) a green and a blue in any order
  - iii) no whites.

Tree diagrams for unequal probabilities

In each of the cases considered so far, all of the outcomes have been assumed to be equally likely. However, this need not be the case.

Worked example In winter, the probability that it rains on any one day is $\frac{5}{7}$.

i) Using a tree diagram, show all the possible combinations for two consecutive days. Write each of the probabilities by the sides of the branches.



Note how the probability of each outcome is found by multiplying the probabilities for each of the branches. This is because each outcome is the result of calculating the fraction of a fraction.

ii) Calculate the probability that it will rain on both days.

This is an outcome that is $\frac{5}{7}$ of $\frac{5}{7}$.

$$P(R, R) = \frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$$

iii) Calculate the probability that it will rain on the first day but not the second day.

$$P(R, NR) = \frac{5}{7} \times \frac{2}{7} = \frac{10}{49}$$

iv) Calculate the probability that it will rain on at least one day.

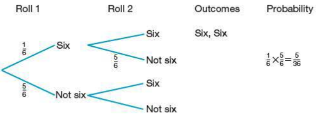
The outcomes which satisfy this event are (R, R) (R, NR) and (NR, R).

Therefore the probability is $\frac{25}{49} + \frac{10}{49} + \frac{10}{49} = \frac{45}{49}$

Exercise 10.4

1. A particular board game involves players rolling a dice. However, before a player can start, he or she needs to roll a 6.

a) Copy and complete the tree diagram below showing all the possible combinations for the first two rolls of the dice.



b) Calculate the probability of the following:
  i) getting a six on the first throw
  ii) starting within the first two throws
  iii) starting on the second throw
  iv) not starting within the first three throws
  v) starting within the first three throws.

c) If you add the answers to Q 1. b (iv) and (v) what do you notice? Explain your answer.

2. In Italy $\frac{3}{5}$ of the cars are made abroad. By drawing a tree diagram and writing the probabilities next to each of the branches, calculate the following probabilities:

a) the next two cars to pass a particular spot are both Italian

b) two of the next three cars are foreign

c) at least one of the next three cars is Italian.

3. The probability that a morning bus arrives on time is 65%.
  a) Draw a tree diagram showing all the possible outcomes for three consecutive mornings.
  b) Label your tree diagram and use it to calculate the probability that:
    i) the bus is on time on all three mornings
    ii) the bus is late the first two mornings
    iii) the bus is on time two out of the three mornings
    iv) the bus is on time at least twice.

4. Light bulbs are packaged in cartons of three. 10% of the bulbs are found to be faulty. Calculate the probability of finding two faulty bulbs in a single carton.

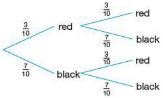
5. A cricket team has a 0.25 chance of losing a game. Calculate the probability of the team achieving:
  a) two consecutive wins
  b) three consecutive wins
  c) ten consecutive wins.

Tree diagrams for probability problems without replacement

In the examples considered so far, the probability for each outcome remained the same throughout the problem. However, this need not always be the case.

Worked examples

a) A bag contains three red balls and seven black balls. If the balls are put back after being picked, what is the probability of picking:
  a) two red balls
  b) a red ball and a black ball in any order.



This is selection with replacement. Draw a tree diagram to help visualise the problem:

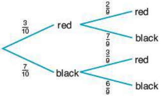
a) The probability of a red followed by a red, P(RR) = $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$.

b) The probability of a red followed by a black or a black followed by a red is

$P(RB) + P(BR) = (\frac{3}{10} \times \frac{7}{10}) + (\frac{7}{10} \times \frac{3}{10}) = \frac{21}{100} + \frac{21}{100} = \frac{42}{100}$.

b) Repeat the previous question, but this time each ball that is picked is not put back in the bag.

This is selection without replacement. The tree diagram is now as shown:



a) $P(RR) = \frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$

b) $P(RB) + P(BR) = \left(\frac{3}{10} \times \frac{7}{9}\right) + \left(\frac{7}{10} \times \frac{3}{9}\right) = \frac{21}{90} + \frac{21}{90} = \frac{42}{90}$.

Exercise 10.5

1. A bag contains five red balls and four black balls. If a ball is picked out at random, its colour recorded and it is then put back in the bag, what is the probability of choosing:
- a) two red balls
- b) two black balls
- c) a red ball and a black ball in this order
- d) a red ball and a black ball in any order?

2. Repeat Q.1 but, in this case, after a ball is picked at random, it is not put back in the bag.

3. A bag contains two black, three white and five red balls. If a ball is picked, its colour recorded and then put back in the bag, what is the probability of picking:
- a) two black balls
- b) a red and a white ball in any order?

4. Repeat Q.3 but, in this case, after a ball is picked at random, it is not put back in the bag.

5. You buy five tickets for a raffle. 100 tickets are sold altogether. Tickets are picked at random. You have not won a prize after the first three tickets have been drawn.
- a) What is the probability that you win a prize with either of the next two draws?
- b) What is the probability that you do not win a prize with either of the next two draws?

6. A normal pack of 52 cards is shuffled and three cards are picked at random. Draw a tree diagram to help calculate the probability of picking:
- a) two clubs first
- b) three clubs
- c) no clubs
- d) at least one club.

7. A bowl of fruit contains one mango, one banana, two oranges and two papayas. Two pieces of fruit are chosen at random and eaten.

a) Draw a tree diagram showing all the possible combinations of the two pieces of fruit.

b) Use your tree diagram to calculate the probability that:

   i) both the pieces of fruit eaten are oranges

   ii) a mango and a banana are eaten

   iii) at least one papaya is eaten.

SECTION 4 Use of Venn diagrams in probability

You saw in Topic 9 how Venn diagrams can be used to represent sets. They can also be used to solve problems involving probability.

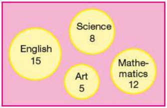
Worked examples

a) In a survey carried out in a college, students were asked which was their favourite subject.

15 chose English
8 chose Science
12 chose Mathematics
5 chose Art

What is the probability that a student chosen at random will like Science the best?

This can be represented on a Venn diagram as:

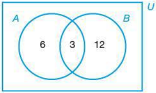


There are 40 students, so the probability is $\frac{8}{40} = \frac{1}{5}$.

b) A group of 21 friends decide to go out for the day to the local town. 9 of them decide to see a film at the cinema and 15 of them go for lunch.

i) Draw a Venn diagram to show this information if set A represents those who see a film and set B represents those who have lunch.

$9 + 15 = 24$; as there are only 21 people, this implies that 3 people see the film and have lunch. This means that $9 - 3 = 6$ only went to see a film and $15 - 3 = 12$ only had lunch.



ii) Determine the probability that a person picked at random only went to the cinema.

The number who only went to the cinema is 6, therefore the probability is $\frac{6}{21} = \frac{2}{7}$.

Exercise 10.6

1. In a class of 30 students, 20 study French, 18 study Spanish and 5 neither.

a) Draw a Venn diagram to show this information.

b) What is the probability that a student chosen at random studies both French and Spanish?

2. In a group of 35 students, 19 take Physics, 18 take Chemistry and 3 take neither. What is the probability that a student chosen at random takes:

a) both Physics and Chemistry

b) Physics only

c) Chemistry only.

3. 108 people visited an art gallery. 60 liked the pictures, 53 liked the sculpture, 10 liked neither.
What is the probability that a person chosen at random, liked the pictures but not the sculpture?

4. In a series of examinations in a school:

37 students took English
48 students took French
45 students took Spanish
15 students took English and French
13 students took French and Spanish
7 students took English and Spanish
5 students took all three.

a) Draw a Venn diagram to represent this information.

b) What is the probability that a student picked at random took:

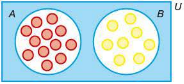
i) all three ii) English only iii) French only.

Laws of probability

Mutually exclusive events

Events that cannot happen at the same time are known as mutually exclusive events. For example, if a sweet bag contains 12 red sweets and 8 yellow sweets let picking a red sweet be event A, whilst picking a yellow sweet be event B. If one sweet is picked, it is not possible to pick a sweet which is both red and yellow. Therefore these events are **mutually exclusive**.

This can be shown in a Venn diagram:



$\mathrm{P}(A) = \frac{12}{20}$ whilst $\mathrm{P}(B) = \frac{8}{20}$.

As there is no overlap, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) = \frac{12}{20} + \frac{8}{20} = \frac{20}{20} = 1$, i.e. the probability of mutually exclusive event A or event B happening is equal to the sum of the probability of event A and event B and the sum of the probabilities of all possible mutually exclusive events is 1.

Worked example

In a 50 m swim, the world record holder has a probability of 0.72 of winning. The probability of her finishing second is 0.25.

What is the probability that she either wins or comes second?

Since she cannot finish both 1st and 2nd, the events are mutually exclusive.

Therefore $\mathrm{P}(1\text{st} \cup 2\text{nd}) = 0.72 + 0.25 = 0.97$.

Combined events

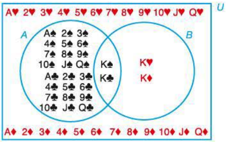
If events are not mutually exclusive, then they may occur at the same time.

These are known as **combined events**.

For example, a pack of 52 cards contains four suits: clubs (♣), spades (♠), hearts (♥) and diamonds (♦). Clubs and spades are black; hearts and diamonds are red. Each suit contains 13 cards. These are ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King.

A card is picked at random. Event A represents picking a black card; event B represents picking a King.

In a Venn diagram this can be shown as:



P(A) $\frac{26}{52} = \frac{1}{2}$ and P(B) $\frac{4}{52} = \frac{1}{13}$

However P($A \cup B$) $\neq \frac{26}{52} + \frac{4}{52}$ because K♠ and K♣ belong to both events A and B and have therefore been counted twice. This is shown in the overlap of the Venn diagram and needs to be taken into account.

Extension

Therefore, for combined events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. the probability of event A or B is equal to the sum of the probabilities of A and B minus the probability of A and B.

Worked example In a holiday survey of 100 people:

72 people have had a beach holiday
16 have had a sightseeing holiday
12 have had both

What is the probability that one person chosen at random from the survey has had either a beach holiday (B) or a sightseeing holiday (S)?

$P(B) = \frac{72}{100}$ $\qquad$ $P(S) = \frac{16}{100}$ $\qquad$ $P(B \cap S) = \frac{12}{100}$

Therefore $P(B \cup S) = \frac{72}{100} + \frac{16}{100} - \frac{12}{100} = \frac{76}{100}$

Independent events

A student may be born on 1 June, another student in his class may also be born on 1 June. These events are independent of each other (assuming they are not twins).

If a dice is thrown and a coin spun, the outcomes of each are also independent, i.e. the outcome of one does not affect the outcome of another.

For independent events, the probability of both occurring is the product of each occurring separately, i.e.

$$P(A \cap B) = P(A) \times P(B)$$

Worked examples

a) You spin a coin and roll a dice.

i) What is the probability of getting a head on the coin and a five on the dice?

$$\mathrm{P(H)} = \tfrac{1}{2} \qquad \mathrm{P(5)} = \tfrac{1}{6}$$

Both events are independent therefore

$$\begin{aligned}\mathrm{P(H \cap 5)} &= \mathrm{P(H)} \times \mathrm{P(5)}\\ &= \tfrac{1}{2} \times \tfrac{1}{6}\\ &= \tfrac{1}{12}\end{aligned}$$

ii) What is the probability of getting either a head on the coin or a five on the dice, but not both.

$\mathrm{P(H \cup 5)}$ is the probability of getting a head, a five or both.

Therefore $\mathrm{P(H \cup 5)} - \mathrm{P(H \cap 5)}$ removes the probability of both events occurring. The solution is

$$\begin{aligned}\mathrm{P(H \cup 5)} - \mathrm{P(H \cap 5)} &= \mathrm{P(H)} + \mathrm{P(5)} - \mathrm{P(H \cap 5)}\\ &= \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{12}\\ &= \tfrac{7}{12}\end{aligned}$$

b) The probabilities of two events X and Y are given by:

$\mathrm{P}(X) = 0.5$, $\mathrm{P}(Y) = 0.4$, and $\mathrm{P}(X \cap Y) = 0.2$.

i) Are events X and Y mutually exclusive?

No: if the events were mutually exclusive, then $\mathrm{P}(X \cap Y)$ would be 0 as the events could not occur at the same time.

ii) Calculate $\mathrm{P}(X \cup Y)$.

$$\begin{aligned}\mathrm{P}(X \cup Y) &= \mathrm{P}(X) + \mathrm{P}(Y) - \mathrm{P}(X \cap Y)\\ &= 0.5 + 0.4 - 0.2\\ &= 0.7\end{aligned}$$

iii) What kind of events are X and Y?

Since $\mathrm{P}(X \cap Y) = \mathrm{P}(X) \times \mathrm{P}(Y)$,

i.e. $0.2 = 0.5 \times 0.4$, events X and Y must be independent.

Probability from contingency tables

A contingency table is a two-way table containing frequency data. The data allows the probabilities of events to be calculated.

Worked example

An icecream seller keeps a record of the number of different types of icecreams he sells to adults and children. The results are displayed in the contingency table below:

	Adults	Children
Vanilla	24	8
Strawberry	15	14
Chocolate	6	28

A person was seen buying one of the icecreams. Calculate the probability of each of the following:

i) It was a child.

Total number of icecreams sold: $24 + 15 + 6 + 8 + 14 + 28 = 95$

Total number of children served: $8 + 14 + 28 = 50$

Therefore $\text{P(Child)} = \frac{50}{95} = \frac{10}{19}$

ii) The icecream being bought was chocolate.

Total number of chocolate icecreams bought: $28 + 6 = 34$

Therefore $\text{P(Chocolate)} = \frac{34}{95}$

iii) It was an adult buying a vanilla icecream.

Number of adults buying vanilla icecreams: 24

Therefore $\text{P(Adult buying vanilla)} = \frac{24}{95}$

Exercise 10.7

1. In a 50 m swim, the record holder has a probability of 0.68 of winning and a 0.25 probability of finishing second. What is her probability of finishing in the first two?

2. The Jamaican 100 m women's relay team has a 0.5 chance of coming first in the final, 0.25 chance of coming second and 0.05 chance of coming third. What is the team's chance of a medal?

3. You spin a coin and throw a dice. What is the probability of getting:

a) a head and a factor of 3
b) a head or a factor of 3
c) a head or a factor of 3, but not both?

4. What is the probability that two people, picked at random, both have a birthday in June?

5. Amelia takes two buses to work. On a particular day, the probability of her catching the first bus is 0.7 and the probability of catching the second bus is 0.5.
The probability of her catching neither is 0.1. If A represents catching the first bus and B the second:
a) state $\mathrm{P}(A \cup B)'$
b) find $\mathrm{P}(A \cup B)$.
c) Given that $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, calculate $\mathrm{P}(A \cap B)$.

6. The probability of Marco having breakfast is 0.75.
The probability that he gets a lift to work is 0.9 if he has had breakfast and 0.8 if he has not.
a) What is the probability of Marco having breakfast then getting a lift?
b) What is the probability of Marco not having breakfast then getting a lift?
c) What is the probability that Marco gets a lift?

7. The numbers and types of books on a student's bookshelf are given in the contingency table below:

	Hardback	**Paperback**
Fiction	8	46
Non-fiction	22	14

Calculate the probability that a book picked from the shelf at random is:
a) a paperback book
b) a fiction book
c) a non-fiction hardback.

8. One morning there are three exams occurring at the same time. The number of boys and girls sitting each exam are given in the contingency table below:

	Maths	**French**	**Philosophy**
Boys	48	6	12
Girls	52	16	8

Calculate the probability that a student picked at random is:
a) a girl
b) sitting the French exam
c) a boy sitting the Philosophy exam.

9. A shirt manufacturer produces a style of shirt in three different colours (white, blue and grey) and in four different sizes (XL, L, M and S).

	White	**Blue**	**Grey**
XL	6	4	8
L	10	20	6
M	12	13	8
S	4	7	2

A shop receives a large box with a delivery of these shirts. The contents of the box are summarised in the contingency table on the left.

a) If the shirts are unpacked in a random order, calculate the probability that the first shirt unpacked is:
  i) white
  ii) large
  iii) large and white.

b) Calculate the probability that the first two shirts unpacked (without replacement) are:
  i) blue
  ii) the same colour
  iii) not all the same colour.

10. a) How many students are in your class?

b) How likely do you think it is that two people in your class will share the same birthday? Very likely? Likely? Approx 50–50? Unlikely? Very unlikely?

c) Write down everybody's birthday. Did two people have the same birthday?

Below is a way of calculating the probability that two people have the same birthday depending how many people there are. To study this, it is easiest to look at the probability of birthday's being *different*. When this probability is less than 50%, then the probability that two people will have the same birthday is greater than 50%.

When the first person asks the second person, the probability of them *not* having the same birthday is $\frac{364}{365}$ (i.e. it is $\frac{1}{365}$ that they have the same birthday).

When the next person is asked, as the events are independent, the probability of all three having different birthdays is:

$$\left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) = 99.2\%$$

When the next person is asked, the probability of all four having different birthdays is:

$$\left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \times \left(\frac{362}{365}\right) = 98.4\%$$

and so on …

d) Copy and complete the table below until the probability is 50%.

Number of people	Probability of them *not* having the same birthday
2	$\left(\frac{364}{365}\right) = 99.7\%$
3	$\left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) = 99.2\%$
4	$\left(\frac{364}{365}\right) \times \left(\frac{363}{365}\right) \times \left(\frac{362}{365}\right) = 98.4\%$
5	
10	
15	
20	
etc	

e) Explain in words what your solution to Q.10 (d) means.

SECTION 6

Experimental probability

So far the work covered has dealt with theoretical probability. However, there are many occasions when the probability of an outcome is not initially known and therefore experiments are carried out in order to make predictions. This is known as **experimental probability**.

For example, a six-sided dice is known to be biased, i.e. not all numbers are equally likely.

The dice is rolled 60 times and the results recorded in the table below:

Number	1	2	3	4	5	6
Frequency	3	7	12	8	20	10

In order to calculate the experimental probability of each number, its **relative frequency** is calculated. The relative frequency refers to the fraction of an amount as a proportion of the total.

Therefore the relative frequency of each number is:

Number	1	2	3	4	5	6
Frequency	3	7	12	8	20	10
Relative frequency	$\frac{3}{60}$	$\frac{7}{60}$	$\frac{12}{60}$	$\frac{8}{60}$	$\frac{20}{60}$	$\frac{10}{60}$

Predictions can be made about future outcomes based on these results.

Worked examples Using the results above calculate the following:

a) If the dice were rolled 100 times, approximately how many times would you expect to get a three?

$P(3) = \frac{12}{60} = \frac{1}{5}$

Therefore the number of threes would be approximately $\frac{1}{5} \times 100 = 20$.

b) The dice was rolled many times. The number two was rolled 38 times. Predict how many times the dice was rolled.

$P(2) = \frac{7}{60}$

Let x be the number of times the dice was rolled:

$$x \times \frac{7}{60} = 38$$

$$\Rightarrow x = \frac{38 \times 60}{7}$$

$$\Rightarrow x = 325.71 \text{ (2 d.p.)}$$

Therefore the dice was rolled approximately 326 times.

Accuracy is improved the more times the experiment is carried out, as any rogue results have a relatively smaller effect. So to improve the accuracy of the results, simply increase the number of trials.

Exercise 10.8

1. A dice is rolled 100 times and the results recorded in the table below:

Number	1	2	3	4	5	6
Frequency	18	15	16	17	16	18

a) Explain, giving reasons, whether you think the dice is fair or biased.

Another dice is rolled 100 times and the results recorded below:

Number	1	2	3	4	5	6
Frequency	10	25	32	5	20	8

b) Explain, giving reasons, whether you think this dice is fair or biased.
c) Calculate the relative frequency of each number on the second dice.
d) If the second dice was rolled 450 times, how many times would you expect to get a six?
e) The second dice was rolled, x times. The number 4 was obtained 23 times. Estimate the value of x.
f) Both dice were rolled 350 times. How many **more** sixes would you expect to get with the first dice compared with the second.

2. A bird spotter wants to find out the likelihood of different types of birds landing in his garden so that he can put out appropriate feed. He conducts a survey over a period of five hours. The results are shown below:

Type of bird	Sparrow	Starling	Crow	Wren	Other
Frequency	46	32	9	16	27

a) Assuming conditions are similar the following day, estimate the number of crows he is likely to spot in a three hour period.
b) The day after he conducts a similar survey and counts 50 starlings.
Approximately how long was he recording results on this occasion?
c) Six months later he decides to estimate the number of sparrows visiting his garden in a two hour period. Explain, giving reasons, whether he should use the original data or not for his estimate.

3. I go to work by bus each day. If the bus is on time, I get to work on time. Over a 20 day period I record whether I arrive at work on time or whether I arrive late. If I arrive late, I also record how late I am. The results are shown below:

Arrival time	On time	5 mins late	10 mins late	15 mins late	20 mins late
Frequency	12	2	1	4	1

a) I work 230 days in a working year.
  i) Estimate how many times I would arrive on time.
  ii) What assumptions have you made in estimating the answer to (i)?
b) Estimate the number of times I would arrive 20 minutes late in a working year.
c) Estimate the total amount of time in hours I arrive late in a working year.
d) In the same city there are 250 000 people who use the buses to get to work each day.
  i) Estimate the total amount of time lost in the city due to late buses.
  ii) What assumptions have you made in estimating the above answer?

4. Check the bias of a dice in your classroom by conducting an experiment.

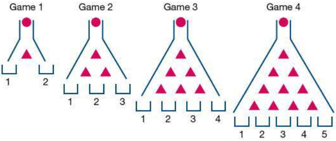
Explain your methods and display your results clearly. Refer to your results when deciding whether or not the dice is biased.

5. Drawing pins, when dropped, can either land point up or point down as shown below:

a) Carry out an experiment to determine the probability of a drawing pin landing point up.
b) What factors are likely to influence whether a drawing pin lands point up or point down?
c) By considering one of the factors you stated in, Q.2 (b) carry out a further experiment to determine whether it does influence how a drawing pin lands.

Probability drop

A game involves dropping a red marble down a chute.
On hitting a triangle divider, the marble can either bounce left or right. On completing the drop, the marble lands in one of the trays along the bottom. The trays are numbered from left to right. Different sizes of game exist, the four smallest versions are shown below:



To land in tray 2 in the second game above, the ball can travel in one of two ways. These are: Left – Right or Right – Left.

This can be abbreviated to LR or RL.

1. State the different routes the marble can take to land in each of the trays in the third game.
2. State the different routes the marble can take to land in each of the trays in the fourth game.
3. State, giving reasons, the probability of a marble landing in tray 1 of the fourth game.
4. State, giving reasons, the probability of a marble landing in each of the other trays in the fourth game.
5. Investigate the probability of the marble landing in each of the different trays of larger games.
6. Using your findings from your investigation, predict the probability of a marble landing in tray 7 of the tenth game (11 trays at the bottom).
7. Investigate the links between this game and the sequence of numbers generated in Pascal's triangle.

Extension

The following question is beyond the scope of the syllabus but is an interesting extension.

8. Investigate the links between this game, Pascal's triangle and the binomial expansion.

Dice sum

Two ordinary dice are rolled and their scores added together.

Below is an incomplete table showing the possible outcomes:

		Dice 1					
		1	2	3	4	5	6
Dice 2	1	2			5		
	2						
	3				7		
	4				8		
	5				9	10	11
	6						12

1. Copy and complete the table to show all possible outcomes.
2. How many possible outcomes are there?
3. What is the most likely total when two dice are rolled?
4. What is the probability of getting a total score of 4?
5. What is the probability of getting the most likely total?
6. How many times more likely is a total score of 5 compared with a total score of 2?

Now consider rolling two four-sided dice each numbered 1–4. Their scores are also added together.

7. Draw a table to show all the possible outcomes when the two four-sided dice are rolled.
8. How many possible outcomes are there?
9. What is the most likely total?
10. What is the probability of getting the most likely total?
11. Investigate the number of possible outcomes, the most likely total and its probability when two identical dice are rolled together and their scores added, i.e. consider 8-sided dice, 10-sided dice, etc.
12. Consider two m-sided dice rolled together and their scores added.
    a) What is the total number of outcomes in terms of m?
    b) What is the most likely total, in terms of m?
    c) What, in terms of m, is the probability of the most likely total.
13. Consider an m-sided and n-sided dice rolled together, where $m > n$.
    a) In terms of m and n, deduce the total number of outcomes.
    b) In terms of m and/or n, deduce the most likely total(s).
    c) In terms of m and/or n, deduce the probability of getting the most likely total.

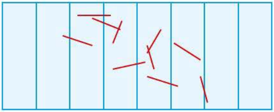
ICT Activity: Buffon's needle experiment

You will need to use a spreadsheet for this activity.

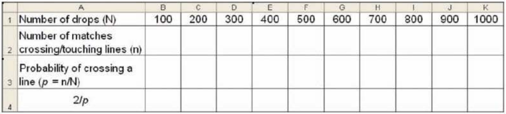
The French count Le Comte de Buffon devised the following probability experiment.

1. Measure the length of a match (with the head cut off) as accurately as possible.
2. On a sheet of paper draw a series of straight lines parallel to each other. The distance between each line should be the same as the length of the match.
3. Take ten identical matches and drop them randomly on the paper. Count the number of matches that cross or touch any of the lines.

For example in the diagram below, the number of matches crossing or touching lines is six.



4. Repeat the experiment a further nine times, making a note of your results, so that altogether you have dropped 100 matches.
5. Set up a spreadsheet similar to the one shown below and enter your results in cell B2.



	A	B	C	D	E	F	G	H	I	J	K
1	Number of drops (N)	100	200	300	400	500	600	700	800	900	1000
2	Number of matches crossing/touching lines (n)										
3	Probability of crossing a line (p = n/N)										
4	2/p										

6. Repeat 100 match drops again, making a total of 200 drops, and enter cumulative results in cell C2.
7. By collating the results of your fellow students, enter the cumulative results of dropping a match 300–1000 times in cells D2–K2 respectively.

8. Using an appropriate formula, get the spreadsheet to complete the calculations in Rows 3 and 4.

9. Use the spreadsheet to plot a line graph of N against $\frac{2}{p}$.

10. What value does $\frac{2}{p}$ appear to get closer to?

SECTION 8 Student assessments

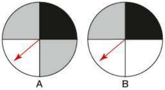
Student assessment 1

1. Calculate the theoretical probability of:
   a) being born on a Saturday
   b) being born on the 5th of a month in a non-leap year
   c) being born on 20 June in a non-leap year
   d) being born on 29 February.

2. When rolling an ordinary fair dice, calculate the theoretical probability of getting:
   a) a 2
   b) an even number
   c) a 3 or more
   d) less than 1.

3. A bag contains 12 white counters, 7 black counters and 1 red counter.
   a) If, when a counter is taken out, it is not replaced, calculate the probability that:
      i) the first counter is white
      ii) the second counter removed is red, given that the first was black.
   b) If, when a counter is picked, it is then put back in the bag, how many attempts will be needed before it is mathematically certain that a red counter will have been picked out?

4. A coin is tossed and an ordinary, fair dice is rolled.
   a) Draw a two-way table showing all the possible combinations.
   b) Calculate the probability of getting:
      i) a head and a six
      ii) a tail and an odd number
      iii) a head and a prime number.



5. Two spinners A and B are split into quarters and coloured as shown. Both spinners are spun.
   a) Draw a fully labelled tree diagram showing all the possible combinations on the two spinners. Write beside each branch the probability of each outcome.
   b) Use your tree diagram to calculate the probability of getting:
      i) two blacks
      ii) two greys
      iii) a grey on spinner A and a white on spinner B.

6. A coin is tossed three times.
   a) Draw a tree diagram to show all the possible outcomes.
   b) Use your tree diagram to calculate the probability of getting:
      i) three tails
      ii) two heads
      iii) no tails
      iv) at least one tail.

7. A goalkeeper expects to save one penalty out of every three. Calculate the probability that he:
   a) saves one penalty out of the next three
   b) fails to save any of the next three penalties
   c) saves two out of the next three penalties.

8. A board game uses a fair dice in the shape of a tetrahedron. The sides of the dice are numbered 1, 2, 3 and 4. Calculate the probability of:
   a) not throwing a 4 in two throws
   b) throwing two consecutive 1s
   c) throwing a total of 5 in two throws.

9. A normal pack of 52 cards is shuffled and three cards picked at random. Calculate the probability that all three cards are picture cards.

Student assessment 2

1. A card is drawn from a standard pack of cards.
   a) Draw a Venn diagram to show the following:
      A is the set of aces
      B is the set of picture cards
      C is the set of clubs
   b) From your Venn diagram find the following probabilities:
      i) P(ace or picture card)
      ii) P(not an ace or picture card)
      iii) P(club or ace)
      iv) P(club and ace)
      v) P(ace and picture card)

2. Students in a school can choose to study one or more science subjects from Physics, Chemistry and Biology.

In a year group of 120 students, 60 took Physics, 60 took Biology and 72 took Chemistry; 34 took Physics and Chemistry, 32 took Chemistry and Biology and 24 took Physics and Biology; 18 took all three.

a) Draw a Venn diagram to represent this information.
b) If a student is chosen at random, what is the probability that:
   i) the student chose to study only one subject
   ii) the student chose Physics or Chemistry and did not choose Biology.

3. A class took an English test and a Mathematics test. 40% passed both tests and 75% passed the English test.

What percentage of those who passed the English test also passed the Mathematics test?

4. A jar contains blue and red counters. Two counters are chosen without replacement. The probability of choosing a blue then a red counter is 0.44. The probability of choosing a blue counter on the first draw is 0.5.

What is the probability of choosing a red counter on the second draw if the first counter chosen was blue?

5. In a group of children, the probability that a child has black hair is 0.7. The probability that a child has brown eyes is 0.55. The probability that a child has either black hair or brown eyes is 0.85.

What is the probability that a child chosen at random has both black hair and brown eyes?

6. It is not known whether a six-sided dice is biased or not. It is rolled 80 times and the results recorded in the table below:

Number	1	2	3	4	5	6
Frequency	11	12	6	18	16	17

a) Explain, giving reasons, whether you think the dice is fair or biased.
b) Calculate the relative frequency of each number.
c) If the dice was rolled 220 times, estimate the amount of times a 3 would be rolled.
d) The dice was rolled x times. The number 6 was rolled 48 times. Estimate the value of x.

TOPIC

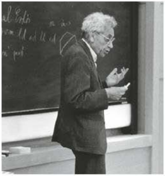
11 Statistics

This topic will cover the following syllabus content:

11.2 Discrete and continuous data
11.3 (Compound) bar chart, line graph, pie chart, stem-and-leaf plot, scatter diagram
11.4 Mean, mode, median, quartiles, range from lists of discrete data
Mean, mode, median and range from grouped discrete data
11.5 Mean from continuous data
11.6 Histograms with frequency density on the vertical axis
11.7 Cumulative frequency table and curve
Median, quartiles, percentiles and inter-quartile range read from curve
11.8 Use of a graphics calculator to calculate mean, median, and quartiles for discrete data and mean for grouped data
11.9 Understanding and description of correlation with reference to a scatter diagram
Straight line of best fit (by eye) through the mean on a scatter diagram
Equation of the linear regression line from a graphics calculator

Sections

SECTION 1 Paul Erdös



Paul Erdös (1913–1996)

Paul Erdös was born in Hungary in 1913. He was a very friendly mathematician. He worked on more than 1500 mathematical papers with more than 500 other mathematicians. The amount of work that he did on graph theory, set theory and probability compares well with the amount of work done by Gauss.

He was a great problem solver and was very good at helping other mathematicians who were stuck with an area of study. Because of his help with their problems, his friends created the Erdös number as a humorous tribute.

Erdös was assigned the Erdös number of 0 (for being himself).

Those who worked directly with him have an Erdös number of 1.

Those who worked with them have an Erdös number of 2, and so on.

It is estimated that 90 percent of the world's mathematicians have an Erdös number smaller than 8.

SECTION 2 Basic graphs and charts

Discrete and continuous data

Discrete data can only take specific values, for example the number of tickets sold for a concert can only be positive integer values.

Continuous data, on the other hand, can take any value within a certain range, for example the time taken to run 100m will typically fall in the range 10–20 seconds. Within that range, however, the time stated will depend on the accuracy required. So a time stated as 13.8s could have been 13.76s, 13.764s or 13.7644s, etc.

Exercise 11.1 State whether the data below is discrete or continuous.

1. Your shoe size

2. Your height

3. Your house number

4. Your weight

5. The total score when two dice are thrown

6. A mathematics exam mark
7. The distance from the Earth to the moon
8. The number of students in your school
9. The speed of a train
10. The density of lead

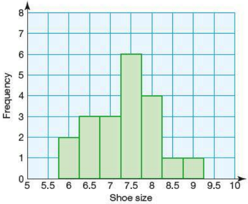
Displaying simple discrete data

Data can be displayed in many different ways. It is therefore important to choose the method that displays the data most clearly and effectively.

The frequency table shows the British shoe sizes of 20 students in a class.

Shoe size	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$	9
Frequency	2	3	3	6	4	1	1

This can be displayed as a **frequency histogram**.



Shoe sizes are an example of discrete data as the data can only take certain values. As a result the frequency histogram has certain properties.

- Each bar is of equal width and its height represents the frequency.
- The bars touch (this is not the case with a bar chart).
- The value is written at the mid-width of each bar. This is because students with a foot size in the range 6.75 – 7.25, for example, would have a shoe size of 7.

Exercise 11.2

1. The figures in the list below give the total number of chocolate sweets in each of 20 packets of the sweets.

   35, 36, 38, 37, 35, 36, 38, 36, 37, 35,
   36, 36, 38, 36, 35, 38, 37, 38, 36, 38

   a) Present the data in a tally and frequency table.
   b) Present the data as a frequency histogram.

2. Record the shoe sizes of everybody in your class.
   a) Present the results in a tally and frequency table.
   b) Present the data as a frequency histogram.
   c) What conclusions can you draw from your results?

Grouped discrete or continuous data

If there is a big range in the data, it is sometimes easier and more useful to group the data in a grouped frequency table. The discrete data below shows the scores for the first round of a golf competition.

71 75 82 96 83 75 76 82 103 85 79 77 83 85 88
104 76 77 79 83 84 86 88 102 95 96 99 102 75 72

Score	Frequency
71–75	5
76–80	6
81–85	8
86–90	3
91–95	1
96–100	3
101–105	4

One possible way of grouping this data in a grouped frequency table is shown on the right.

Note: The groups are arranged so that no score can appear in two groups.

Each group has an upper and lower bound. The lower bound of a group is the smallest possible number that would round up to that group and the upper bound is the highest possible number that would round down to the group, for example for the group 101–105, the lower bound is 100.5 and the upper bound 105.5.

Exercise 11.3

1. The following data gives the percentage scores obtained by students from two classes, 11X and 11Y, in a Mathematics exam.

   11X
   42 73 93 85 68 58 33 70 71 85 90 99 41 70 65
   80 73 89 88 93 49 50 57 64 78 79 94 80 50 76 99

   11Y
   70 65 50 89 96 45 32 64 55 39 45 58 50 82 84
   91 92 88 71 52 33 44 45 53 74 91 46 48 59 57 95

a) Draw a grouped tally and frequency table for each of the classes.
b) Comment on any similarities or differences between the results.

2. The number of apples collected from 50 trees is recorded below:

35 78 15 65 69 32 12 9 89 110 112 148 98
67 45 25 18 23 56 71 62 46 128 7 133 96
24 38 73 82 142 15 98 6 123 49 85 63 19
111 52 84 63 78 12 55 138 102 53 80

Choose suitable groups for this data and represent it in a grouped frequency table.

With grouped continuous data, the groups are presented in a different way.

The results below are the times given (in h:min:s) for the first 50 people completing a marathon.

2:07:11	2:08:15	2:09:36	2:09:45	2:10:45
2:10:46	2:11:42	2:11:57	2:12:02	2:12:11
2:13:12	2:13:26	2:14:26	2:15:34	2:15:43
2:16:25	2:16:27	2:17:09	2:18:29	2:19:26
2:19:27	2:19:31	2:20:00	2:20:23	2:20:29
2:21:47	2:21:52	2:22:32	2:22:48	2:23:08
2:23:17	2:23:28	2:23:46	2:23:48	2:23:57
2:24:04	2:24:12	2:24:15	2:24:24	2:24:29
2:24:45	2:25:18	2:25:34	2:25:56	2:26:10
2:26:22	2:26:51	2:27:14	2:27:23	2:27:37

The data can be arranged into a grouped frequency table as follows:

Group	Frequency
$2{:}05{:}00 \leqslant t < 2{:}10{:}00$	4
$2{:}10{:}00 \leqslant t < 2{:}15{:}00$	9
$2{:}15{:}00 \leqslant t < 2{:}20{:}00$	9
$2{:}20{:}00 \leqslant t < 2{:}25{:}00$	19
$2{:}25{:}00 \leqslant t < 2{:}30{:}00$	9

Note that, as with discrete data, the groups do not overlap. However, as the data is continuous, the groups are written using inequalities. The first group includes all times from 2 h 5 min *up to but not including* 2 h 10 min.

With continuous data, the upper and lower bound of each group are the numbers written as the limits of the group. In the example above, for the group $2{:}05{:}00 \leqslant t < 2{:}10{:}00$, the lower bound is 2:05:00; the upper bound is considered to be 2:10:00 despite it not actually being included in the inequality.

Pie charts

Pie charts are a popular way of displaying data clearly. A pie chart consists of a circle divided into sectors where the angle of each sector is proportional to the relative size of the quantity it represents.

The table below shows the number of goals scored by a football team (A) over the 40 games it played during one season.

Number of goals	0	1	2	3	4
Frequency	22	8	6	3	1

The table shows that Team A scored no goals in 22 of their matches, 1 goal in 8 of their matches.

As the angle at the centre of a circle is 360°, each frequency must be converted to a fraction out of 360 as shown:

Number of goals	Frequency	Relative frequency	Angle
0	22	$\frac{22}{40}$	$\frac{22}{40} \times 360 = 198$
1	8	$\frac{8}{40}$	$\frac{8}{40} \times 360 = 72$
2	6	$\frac{6}{40}$	$\frac{6}{40} \times 360 = 54$
3	3	$\frac{3}{40}$	$\frac{3}{40} \times 360 = 27$
4	1	$\frac{1}{40}$	$\frac{1}{40} \times 360 = 9$

The pie chart can now be drawn using a protractor.

Compound bar chart

A compound bar chart is similar to a pie chart as it also displays the relative proportion of each quantity. However, instead of using a circle, it presents the data in a single column/bar. Each quantity is expressed as a percentage of the total with the height of the bar representing 100%.

Using the previous example of Team A's goals, the relative frequency table can be adapted to percentages as shown:

Number of goals	Frequency	Relative frequency	Percentage
0	22	$\frac{22}{40}$	$\frac{22}{40} \times 100 = 55$
1	8	$\frac{8}{40}$	$\frac{8}{40} \times 100 = 20$
2	6	$\frac{6}{40}$	$\frac{6}{40} \times 100 = 15$
3	3	$\frac{3}{40}$	$\frac{3}{40} \times 100 = 7.5$
4	1	$\frac{1}{40}$	$\frac{1}{40} \times 100 = 2.5$

Compound bar charts are particularly useful when comparisons need to be made between different sets of data.

Worked example Another football team (B) in the same league as Team A has the following goal-scoring record over the same season.

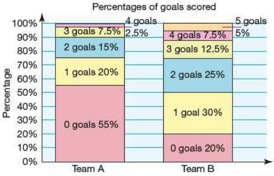
Number of goals	0	1	2	3	4	5
Frequency	8	12	10	5	3	2

a) Draw a compound bar chart for Team B on the same axes as the one for Team A.

A relative frequency and percentage table can be calculated to give:

Number of goals	Frequency	Relative frequency	Percentage
0	8	$\frac{8}{40}$	$\frac{8}{40} \times 100 = 20$
1	12	$\frac{12}{40}$	$\frac{12}{40} \times 100 = 30$
2	10	$\frac{10}{40}$	$\frac{10}{40} \times 100 = 25$
3	5	$\frac{5}{40}$	$\frac{5}{40} \times 100 = 12.5$
4	3	$\frac{3}{40}$	$\frac{3}{40} \times 100 = 7.5$
5	2	$\frac{2}{40}$	$\frac{2}{40} \times 100 = 5$

Drawing both compound charts on the same axes means the results of both teams can be compared more easily.



b) From the charts, decide which of the two teams is likely to be doing better in the league. Justify your answer.

The bar charts suggest that Team B is the more successful team as it scores no goals less often than Team A and also has a higher proportion of goals scored than Team A.

c) Explain why your conclusion to part (c) may be incorrect.

The chart only displays the number of goals scored and does not show the number of goals conceded. Team B may be better than Team A at scoring goals but it may also let in more goals.

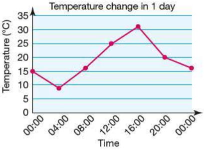
Line graphs

A line graph is a good way of analysing data over a period of time. Data is collected and plotted as coordinates on a graph and a line is then drawn passing through each of the points.

The table shows the temperature taken at four-hourly intervals during one day in New York.

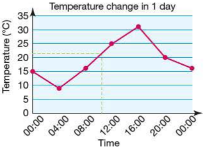
Time	00:00	04:00	08:00	12:00	16:00	20:00	24:00
Temperature (°C)	15	9	16	25	31	20	16

Plotting a line graph shows the changes in temperature during the day:



As time is continuous, it is mathematically acceptable to draw a line passing through each of the points. It is a valid assumption to make that the temperature changes at a constant rate between each of the readings taken. This means that predictions can be made about the temperature at times of the day when readings were not actually taken.

For example, to estimate the temperature at 10:00, a line is drawn up from the horizontal axis at 10:00 until it meets the graph. A line is then drawn horizontally until it reaches the vertical axis, where the temperature can be read.



The temperature at 10:00 was approximately 21°C.

Exercise 11.4

1. A teenager decides to keep a record of her spending habits over a period of a month. Her results are displayed in the table below:

Category	Amount spent ($)
Clothes	110
Entertainment	65
Food	40
Transport	20
Gifts	20
Other	15

a) Calculate the angle represented by each category for the purpose of drawing a pie chart.
b) Draw a pie chart representing the teenager's spending habits.

2. The brother of the teenager in Q.1 also keeps a record of his spending over the same period of time. His results are shown below:

Category	Amount spent ($)
Clothes	10
Entertainment	150
Food	10
Transport	20
Gifts	5
Other	50

a) Find each of these amounts as a percentage of the total.
b) Draw a compound bar chart of his data.

SECTION 3

Stem-and-leaf plots

Stem-and-leaf plots (or stem-and-leaf diagrams) are a special type of bar chart, in which the bars are made from the data itself. This has the advantage that the original data can be recovered easily from the diagram. The stem is the first digit of the numbers, so if the numbers are 63, 65, 67, 68, 69, the stem is 6. The leaves are the remaining numbers written in order.

Worked example The ages of people on a coach transferring them from an airport to a ski resort are as follows:

22	24	25	31	33	23	24	26	37	42
40	36	33	24	25	18	20	27	25	33
28	33	35	39	40	48	27	25	24	29

Display the data on a stem-and-leaf diagram.

1	8
2	0 2 3 4 4 4 4 5 5 5 5 6 7 7 8 9
3	1 3 3 3 3 5 6 7 9
4	0 0 2 8

Key 2 | 5 means 25

Note that the key states what the stem means. If the data were 1.8, 2.7, 3.2 etc, the key would state that 2 | 7 means 2.7.

Exercise 11.5

1. A test in Mathematics is marked out of 40. The scores for the class of 32 students are shown below:

24	27	30	33	26	27	28	39
21	18	16	33	22	38	33	21
16	11	14	23	37	36	38	22
28	15	9	17	28	33	36	34

Display the data on a stem-and-leaf diagram.

2. A basketball team played 24 matches in the 2010 season. Their scores are shown below:

62	48	85	74	63	67	71	83
46	52	63	65	72	76	68	58
54	46	88	55	46	52	58	54

Display the scores on a stem-and-leaf diagram.

3. A class of 27 students was asked to draw a line 8 cm long with a straight edge rather than with a ruler. The lines were then measured and their lengths to the nearest millimetre were recorded.

8.8	6.2	8.3	7.9	8.0	5.9	6.2	10.0	9.7
7.9	5.4	6.8	7.3	7.7	8.9	10.4	5.9	8.3
6.1	7.2	8.3	9.4	6.5	5.8	8.8	8.0	7.3

Present this data using a stem-and-leaf diagram.

Back-to-back diagrams

Stem-and-leaf diagrams are often used as an easy way to compare two sets of data. The leaves are usually put 'back-to-back' on either side of the stem.

Worked example

The stem-and-leaf diagram for the ages of people on a coach to a ski resort (as in the previous worked example) is shown below. The data is easily accessible.

Key 2 | 5 means 25

1	8
2	0 2 3 4 4 4 4 5 5 5 5 6 7 7 8 9
3	1 3 3 3 3 5 6 7 9
4	0 0 2 8

A second coach from the airport is taking people to a golfing holiday. The ages of the people are shown below:

43	46	52	61	65	38	36	28	37	45
69	72	63	55	46	34	35	37	43	48
54	53	47	36	58	63	70	55	63	64

Display the two sets of data on a back-to-back stem-and-leaf diagram.

Golf		Skiing
	1	8
8	**2**	0 2 3 4 4 4 4 5 5 5 5 6 7 7 8 9
8 7 7 6 6 5 4	**3**	1 3 3 3 3 5 6 7 9
8 7 6 6 5 3 3	**4**	0 0 2 8
8 5 5 4 3 2	**5**	
9 5 4 3 3 3 1	**6**	
2 0	**7**	

Key 3 | 5 means 35

Exercise 11.6

1. Write three sentences commenting on the back-to-back stem-and-leaf diagram in the worked example above.
2. The basketball team in Q.2 of Exercise 11.5 had replaced their team coach at the end of the 2010 season. Their scores for the 24 matches played in the 2010 season are shown below:

82	32	88	24	105	63	86	42
35	88	78	106	64	72	88	26
35	41	100	48	54	36	28	33

Display the scores from both seasons on a back-to-back stem-and-leaf diagram and comment on it.

3. The Mathematics test results shown in Q.1 of Exercise 11.5 were for test B. Test A had already been set and marked and the teacher had gone over some of the questions with the class. The marks out of 40 for test A are shown below:

22	18	9	11	38	33	21	14
16	8	12	37	39	25	23	18
34	36	23	16	14	12	22	29
33	35	12	17	22	28	32	39

Draw a back-to-back stem-and-leaf diagram for scores from both tests and comment on.

SECTION 4

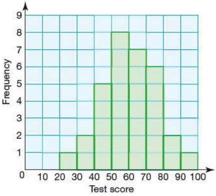
Histograms and frequency density

The frequency histograms introduced in Section 2 all had bars of equal width as in the next example.

Worked example The table shows the marks out of 100 in a Mathematics exam for a class of 32 students. Draw a histogram representing this data.

Test marks	Frequency
1–10	0
11–20	0
21–30	1
31–40	2
41–50	5
51–60	8
61–70	7
71–80	6
81–90	2
91–100	1

All the class intervals are the same. As a result the bars of the histogram will all be of equal width, and the frequency can be plotted on the vertical axis as shown below. Note that the upper and lower bounds of each group are used to draw the bars.

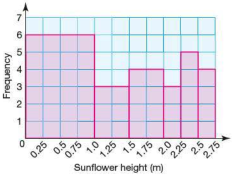


When the class widths are different, the frequency is proportional to the area of the bar and we plot the **frequency density** on the vertical axis.

Worked example The heights of 25 sunflowers were measured and the results recorded in the table.

Height (m)	Frequency
$0 \leqslant h < 1.0$	6
$1.0 \leqslant h < 1.5$	3
$1.5 \leqslant h < 2.0$	4
$2.0 \leqslant h < 2.25$	3
$2.25 \leqslant h < 2.50$	5
$2.50 \leqslant h < 2.75$	4

If a histogram were drawn with frequency plotted on the vertical axis, then it would look like the one shown.



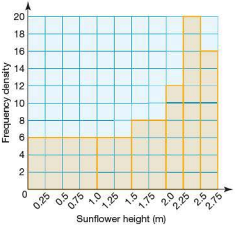
The graph is misleading because it leads people to the conclusion that most of the sunflowers are under 1 m, simply because the area of the bar is so great. In actual fact, only approximately one quarter of the sunflowers were under 1 m. When class intervals are different, it is the area of the bar which represents the frequency not the height. Instead of frequency being plotted on the vertical axis, frequency density is plotted.

$$\text{Frequency density} = \frac{\text{frequency}}{\text{class width}}$$

The results of the sunflower measurements above can therefore be written as:

Height (m)	Frequency	Frequency density
$0 \leqslant h < 1.0$	6	$6 \div 1 = 6$
$1.0 \leqslant h < 1.5$	3	$3 \div 0.5 = 6$
$1.5 \leqslant h < 2.0$	4	$4 \div 0.5 = 8$
$2.0 \leqslant h < 2.25$	3	$3 \div 0.25 = 12$
$2.25 \leqslant h < 2.50$	5	$5 \div 0.25 = 20$
$2.50 \leqslant h < 2.75$	4	$4 \div 0.25 = 16$

The histogram can therefore be redrawn as shown giving a more accurate representation of the data.



Exercise 11.7

1. The table shows the time taken, in minutes, by 40 students to travel to school.

Time (min)	Frequency	Frequency density
$0 \leqslant t < 10$	6	
$10 \leqslant t < 15$	3	
$15 \leqslant t < 20$	13	
$20 \leqslant t < 25$	7	
$25 \leqslant t < 30$	3	
$30 \leqslant t < 40$	4	
$40 \leqslant t < 60$	4	

a) Copy the table and complete it by calculating the frequency density.
b) Show the information on a histogram.

2. Derek and Finn did a survey of the people living in their village. Part of their results are set out in the table.

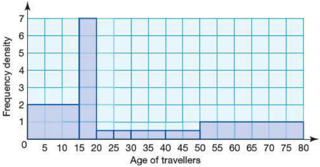
Age (years)	Frequency	Frequency density
$0 \leqslant a < 1$	35	
$1 \leqslant a < 5$		12
$5 \leqslant a < 10$		28
$10 \leqslant a < 20$	180	
$20 \leqslant a < 40$	260	
$40 \leqslant a < 60$		14
$60 \leqslant a < 90$	150	

a) Copy the table and complete it by calculating either the frequency or the frequency density.
b) Show the information on a histogram.

3. The table shows the ages of 150 people, chosen randomly, taking the 6:00a.m. train into a city.

Age (years)	Frequency
$0 \leqslant a < 15$	3
$15 \leqslant a < 20$	25
$20 \leqslant a < 25$	20
$25 \leqslant a < 30$	30
$30 \leqslant a < 40$	32
$40 \leqslant a < 50$	30
$50 \leqslant a < 80$	10

The histogram below shows the results obtained when the same survey was carried out on the 11:00a.m. train.



a) Draw a histogram for the 6:00a.m. train.
b) Compare the two sets of data and give two possible reasons for the differences.

SECTION 5 Averages and ranges

Averages

'Average' is a word which, in general use, is taken to mean somewhere in the middle. For example, a woman may describe herself as being of average height. A student may think that he or she is of average ability in Science. Mathematics is more precise and uses three main methods to measure average.

1. The **mode** is the value occurring most often.
2. The **median** is the middle value when all the data is arranged in order of size.
3. The **mean** is found by adding together all the values of the data and then dividing the total by the number of data values.

Worked example The numbers below represent the number of goals scored by a hockey team in the first 15 matches of the season. Find the mean, median and mode of the goals.

1 0 2 4 1 2 1 1 2 5 5 0 1 2 3

$$\text{Mean} = \frac{1+0+2+4+1+2+1+1+2+5+5+0+1+2+3}{15} = 2$$

Arranging all the data in order and then picking out the middle number gives the median:

0 0 1 1 1 1 1 ② 2 2 2 3 4 5 5

The mode is the number that appears most often. Therefore the mode is 1.

Note: If there is an even number of data values, then there will not be one middle number, but a middle pair. The median is calculated by working out the mean of the middle pair.

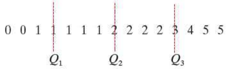
Quartiles and range

Just as the median takes the middle value by splitting the data into two halves, quartiles split the data into quarters.

Taking the example above with the data still arranged in order:

0 0 1 1 1 1 1 2 2 2 2 3 4 5 5

Splitting the data into quarters produces the following:



Q_1 is known as the **lower quartile**, Q_2 as already described is the median and Q_3 is known as the **upper quartile**.

The position of the quartiles can be calculated using simple formulae. For n data values, $Q_1 = \frac{1}{4}(n+1)$ and $Q_3 = \frac{3}{4}(n+1)$.

As with the median, if the position of a quartile falls midway between two data values, then its value is the mean of the two.

Worked example Calculate the lower and upper quartiles of this set of numbers:

7 7 8 12 12 12 15 16 21

Position of $Q_1 = \dfrac{n+1}{4} = \dfrac{10}{4} = 2.5$

Position of $Q_3 = \dfrac{3(n+1)}{4} = \dfrac{30}{4} = 7.5$

The data set can therefore be split as shown below:

Therefore $Q_1 = 7.5$ and $Q_3 = 15.5$.

The **inter-quartile range** is the spread of the middle 50% of the data and can be calculated as the difference between the upper and lower quartiles, i.e.

inter-quartile range = upper quartile – lower quartile.

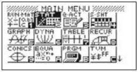
To find the **range** of a data set, simply subtract the smallest data value from the largest data value.

Your graphics calculator is also capable of calculating the mean, median and quartiles of a set of discrete data.

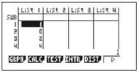
To calculate these for the data set at the start of this section, follow the instructions below:

Casio

SET UP MENU Mat V 2 to select the stat. mode.



Enter the data in List 1.



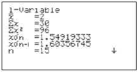
	List 1	List 2	List 3	List 4
SUB				
1	1			
2	0			
3	2			
4	4			

Zoom F2 to access the calculations menu.

G↔T F6 to check the setup.

```
1Var XList  :List1
1Var Freq   :1
2Var XList  :List1
2Var YList  :List2
2Var Freq   :1
```

The data has 1 variable (number of goals), it is in List 1 and each value should be counted once. EXE

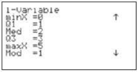


```
1-Variable
x̄      =2
Σx     =30
Σx²    =96
xσn    =1.54919333
xσn-1  =1.60356745
n      =15                ↓
```

Trace F1 to perform the statistical calculations.

The following screen summarises the results of many calculations.

The screen can be scrolled to reveal further results.



```
1-Variable
minX =0                   ↑
Q1   =1
Med  =2
Q3   =3
maxX =5
Mod  =1                   ↓
```

Note: the mean is given by $\bar{x}$ and the median by 'Med'.
The range can be calculated by subtracting 'minX' from 'maxX' and the inter-quartile range by subtracting Q_1 from Q_3.

Texas

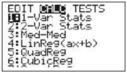
LIST STAT **L1 Y 1** to enter the data into lists.

Enter the data in List 1.

```
L1      L2      L3     1
1       ------  ------
0
2
4
1
2
1
L1(1)=1
```

LIST STAT

to select the 'Calc' menu.

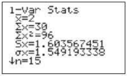


```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
```

L1 Y 1 followed by **2ND** **L1 Y 1** to perform statistical calculations on the 1-variable data in List 1.

```
1-Var Stats L1
```

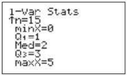
ENTRY SOLVE ENTER



```
1-Var Stats
 x̄=2
 Σx=30
 Σx²=96
 Sx=1.603567451
 σx=1.549193338
↓n=15
```

The following screen summarises the results of many calculations.

The screen can be scrolled to reveal further results.



```
1-Var Stats
↑n=15
 minX=0
 Q₁=1
 Med=2
 Q₃=3
 maxX=5
```

Note: the mean is given by $\bar{x}$ and the median by 'Med'.
The range can be calculated by subtracting 'minX' from maxX' and the inter-quartile range by subtracting Q_1 from Q_3.

Exercise 11.8

1. Find the mean, median, mode, quartiles and range for each set of data.
   a) The number of goals scored by a water polo team in each of 15 matches:
   1 0 2 4 0 1 1 1 2 5 3 0 1 2 2
   b) The total scores when two dice are rolled:
   7 4 5 7 3 2 8 6 8 7 6 5 11 9 7 3 8 7 6 5
   c) The number of students present in a class over a three-week period:
   28 24 25 28 23 28 27 26 27 25 28 28 28 26 25
   d) An athlete's training times (in seconds) for the 100 m race:
   14.0 14.3 14.1 14.3 14.2 14.0 13.9 13.8
   13.9 13.8 13.8 13.7 13.8 13.8 13.8

2. The mean mass of the 11 players in a football team is 80.3 kg. The mean mass of the team plus a substitute is 81.2 kg. Calculate the mass of the substitute.

3. After eight matches a basketball player had scored a mean of 27 points. After three more matches his mean was 29. Calculate the total number of points he scored in the last three games.

Large amounts of data

When there are only 3 sets of data, the median value is given by the second value.

i.e. 1 ② 3.

When there are four values in a set of data, the median value is given by the mean of the second and third values,

i.e. 1 ②③ 4.

When there are five values in a set of data, the median value is given by the third value.

If this pattern is continued, it can be deduced that for *n* values in a set of data, the median value is given by the value at $\frac{n+1}{2}$. This is useful when finding the median of large sets of data.

Worked example The British shoe sizes of 49 people are recorded in the table below. Calculate the median, mean and modal shoe size.

Shoe size	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7
Frequency	2	4	5	9	8	6	6	5	4

As there are 49 data values, the median value is the 25th value. This occurs within shoe size 5:

$2 + 4 + 5 + 9 = 20$ but $2 + 4 + 5 + 9 + 8 = 28$.

So the median shoe size is 5. To calculate the mean shoe size:

$$\frac{(3 \times 2) + (3\tfrac{1}{2} \times 4) + (4 \times 5) + (4\tfrac{1}{2} \times 9) + (5 \times 8) + (5\tfrac{1}{2} \times 6) + (6 \times 6) + (6\tfrac{1}{2} \times 5) + (7 \times 4)}{49} = \frac{250}{49}$$

So the mean shoe size is 5.10 (correct to 3 significant figures).

Note: The mean value is not necessarily a data value which appears in the set or a real shoe size.

The modal shoe size is $4\frac{1}{2}$.

The range of the data is, as before, the smallest data value subtracted from the largest data value. In this case the largest recorded shoe size is 7 and the smallest 3. Therefore the range is 4.

These calculations can also be carried out on your graphics calculator by entering the frequency tables:

Casio

MENU 2 to select the stat. mode.

Enter the shoe sizes in List 1 and their frequency in List 2.

F2 to access the calculations menu.

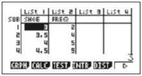
F6 to check the setup.

The data has one variable (shoe size), it is in List 1 and its frequency is in List 2.

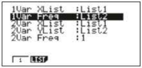
EXE

F1 to perform the statistical calculations.

The screen summarises the results of many calculations and can be scrolled to reveal further results.

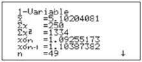


```
    |List 1|List 2|List 3|List 4
SUB |SHOE  |FREQ  |      |
  1 |     3|     2|      |
  2 |   3.5|     4|      |
  3 |     4|     5|      |
  4 |   4.5|     9|      |
GRPH CALC TEST INTR DIST ▷
```

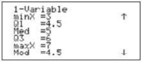


```
1Var XList  :List1
1Var Freq   :List2
2Var XList  :List1
2Var YList  :List2
2Var Freq   :1

1 LIST
```



```
1-Variable
x̄     =5.10204081
Σx    =250
Σx²   =1334
xσn   =1.09255173
xσn-1 =1.10387382
n     =49          ↓
```



```
1-Variable
minX =3            ↑
Q1   =4.5
Med  =5
Q3   =6
maxX =7
Mod  =4.5          ↓
```

Texas

STAT 1 to enter the data into lists.

Enter the shoe size in List 1 and the frequency in List 2.

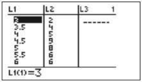
STAT ► to select the 'Calc' menu.

1 followed by

2ND 1 , 2ND 2 to perform statistical calculations on the 1-variable data in List 1 with frequency in List 2.

ENTER

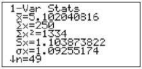
The screen summarises the results of many calculations and can be scrolled to reveal further results.



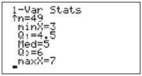
```
L1    |L2   |L3    1
3     |2    |------
3.5   |4    |
4     |5    |
4.5   |9    |
5     |8    |
5.5   |6    |
6     |6    |
L1(1)=3
```

```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
```

```
1-Var Stats L1,L
2
```



```
1-Var Stats
x̄=5.102040816
Σx=250
Σx²=1334
Sx=1.103873822
σx=1.09255174
↓n=49
```



```
1-Var Stats
↑n=49
 minX=3
 Q1=4.5
 Med=5
 Q3=6
 maxX=7
```

Exercise 11.9

1. An ordinary dice was rolled 60 times. The results are shown in the table below. Calculate the mean, median and mode of the scores.

Score	1	2	3	4	5	6
Frequency	12	11	8	12	7	10

2. Two dice were rolled 100 times. Each time their combined score was recorded. Below is a table of the results. Calculate the mean, median and mode of the scores.

Score	2	3	4	5	6	7	8	9	10	11	12
Frequency	5	6	7	9	14	16	13	11	9	7	3

3. Sixty flowering bushes are planted. At their flowering peak, the number of flowers per bush is counted and recorded. The results are shown in the table below:

Flowers per bush	0	1	2	3	4	5	6	7	8
Frequency	0	0	0	6	4	6	10	16	18

a) Calculate the mean, median and mode of the number of flowers per bush.

b) Which of the mean, median and mode would be most useful when advertising the bush to potential buyers?

Mean and mode for grouped data

As has already been described, sometimes it is more useful to group data, particularly if the range of values is very large. However, by grouping data, some accuracy is lost.

The results below are the distances (to the nearest metre) run by twenty students in one minute.

256 271 271 274 275 276 276 277 279 280
281 282 284 286 287 288 296 300 303 308

Table 1: Class interval of 5

Group	250–254	255–259	260–264	265–269	270–274	275–279	280–284	285–289	290–294	295–299	300–304	305–309
Frequency	0	1	0	0	3	5	4	3	0	1	2	1

Table 2: Class interval of 10

Group	250–259	260–269	270–279	280–289	290–299	300–309
Frequency	1	0	8	7	1	3

Table 3: Class interval of 20

Group	250–269	270–289	290–309
Frequency	1	15	4

The three tables above highlight the effects of different group sizes. Table 1 is perhaps too detailed, whilst in Table 3 the group sizes are too big and therefore most of the results fall into one group. Table 2 is the most useful in that the spread of the results is still clear, although detail is lost. In the 270–279 group we can see that there are eight students, but without the raw data, we would not know where in the group they lie.

To find the mean of grouped data, we assume that all the data within a group takes the **mid-interval value**. For example, using Table 2 above,

Group	250–259	260–269	270–279	280–289	290–299	300–309
Mid-interval value	254.5	264.5	274.5	284.5	294.5	304.5
Frequency	1	0	8	7	1	3

$$\text{Estimated mean} = \frac{\begin{array}{c}(254.5 \times 1) + (264.5 \times 0) + (274.5 \times 8) + \\ (284.5 \times 7) + (294.5 \times 1) + (304.5 \times 3)\end{array}}{20} = 282.5$$

The **estimate** of mean distance run is 282.5 metres.
The modal group is 270–279.

Note: In the example above the distance data is rounded to the nearest whole number. This has the effect of presenting continuous data as discrete data. If the data had been truly continous, the groupings would need to be presented differently as shown below.

Group	250–	260–	270–	280–	290–	300–310
Mid-interval value	255	265	275	285	295	305
Frequency	1	0	8	7	1	3

The group 250– in the table above means any result that falls in the group from 250 up to, but not including, 260. It therefore has a group width of 10 rather than 9 as before. The mid-interval values are therefore affected as is the estimate for the mean.

$$\text{Estimated mean} = \frac{\begin{array}{c}(255 \times 1) + (265 \times 0) + (275 \times 8) + \\ (285 \times 7) + (295 \times 1) + (305 \times 3)\end{array}}{20} = 283\,\text{m}$$

The graphics calculator can work out the mean and median of grouped data. The mid-interval value should be entered in List 1 and the frequency in List 2. Then proceed as before.

Exercise 11.10

1. A pet shop has 100 tanks containing fish. The number of fish in each tank is recorded in the table below.

No. of fish	0–9	10–19	20–29	30–39	40–49
Frequency	7	12	24	42	15

a) Calculate an estimate for the mean number of fish in each tank.
b) Give the modal group size.

2. A school has 148 Year 11 students studying Mathematics. Their percentage scores in their Mathematics mock examination are recorded in the grouped frequency table.

% Score	0–9	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89	90–99
Frequency	3	2	4	6	8	36	47	28	10	4

a) Calculate the mean percentage score for the mock examination.
b) What was the modal group score?

3. A stationmaster records how many minutes late each train is. The table of results is shown below:

No. of minutes late	0–	5–	10–	15–	20–	25–30
Frequency	16	9	3	1	0	1

a) Calculate an estimate for the mean number of minutes late a train is.
b) What is the modal number of minutes late?
c) The stationmaster's report concludes: 'Trains are, on average less than five minutes late'. Comment on this conclusion.

SECTION 6

Cumulative frequency

Calculating the cumulative frequency is done by adding up the frequencies as we go along. A cumulative frequency graph is particularly useful when trying to calculate the median of a large set of data, grouped or continuous data, or when trying to establish how consistent a set of results are.

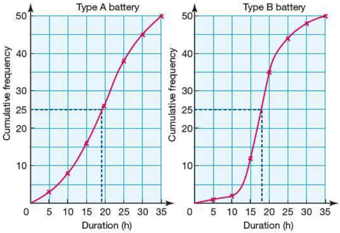
Worked example

The duration of two different brands of battery, A and B, is tested. Fifty batteries of each type are randomly selected and tested in the same way. The duration of each battery is then recorded. The results of the tests are shown in the tables below.

Type A: duration (h)	Frequency	Cumulative frequency
$0 \leqslant t < 5$	3	3
$5 \leqslant t < 10$	5	8
$10 \leqslant t < 15$	8	16
$15 \leqslant t < 20$	10	26
$20 \leqslant t < 25$	12	38
$25 \leqslant t < 30$	7	45
$30 \leqslant t < 35$	5	50

Type B: duration (h)	Frequency	Cumulative frequency
$0 \leqslant t < 5$	1	1
$5 \leqslant t < 10$	1	2
$10 \leqslant t < 15$	10	12
$15 \leqslant t < 20$	23	35
$20 \leqslant t < 25$	9	44
$25 \leqslant t < 30$	4	48
$30 \leqslant t < 35$	2	50

i) Plot a cumulative frequency curve for each brand of battery.



The points are plotted at the upper boundary of each class interval rather than at the middle of the interval. So, for Type A, points are plotted at (5, 3), (10, 8), etc. The points are joined with a smooth curve which is extended to include (0, 0).

ii) Estimate the median duration for each brand.
The median value is the value which occurs half-way up the cumulative frequency axis. This is shown with broken lines on the graph. Therefore:

Median for Type A batteries ≈ 19 h
Median for Type B batteries ≈ 18 h

This tells us that, on average, batteries of Type A last longer (19 hours) than batteries of Type B (18 hours).

Exercise 11.11

1. Sixty athletes enter a cross-country race. Their finishing times are recorded and are shown in the table below:

Finishing time (h)	0–	0.5–	1.0–	1.5–	2.0–	2.5–	3.0–3.5
Frequency	0	0	6	34	16	3	1
Cumulative freq.							

a) Copy the table and calculate the values for the cumulative frequency.
b) Draw a cumulative frequency curve of the results.
c) Show how your graph could be used to find the approximate median finishing time.
d) What does the median value tell us?

2. Three Mathematics classes take the same test in preparation for their final examination. Their raw scores are shown in the table below:

Class A	12, 21, 24, 30, 33, 36, 42, 45, 53, 53, 57, 59, 61, 62, 74, 88, 92, 93
Class B	48, 53, 54, 59, 61, 62, 67, 78, 85, 96, 98, 99
Class C	10, 22, 36, 42, 44, 68, 72, 74, 75, 83, 86, 89, 93, 96, 97, 99, 99

a) Using the class intervals $0 \leqslant x < 20$, $20 \leqslant x < 40$, etc, draw up a grouped frequency table and cumulative frequency table for each class.
b) Draw a cumulative frequency curve for each class.
c) Show how your graph could be used to find the median score for each class.
d) What does the median value tell us?

3. The table below shows the heights of students in a class over a three-year period.

Height (cm)	Frequency 2008	Frequency 2009	Frequency 2010
150–	6	2	2
155–	8	9	6
160–	11	10	9
165–	4	4	8
170–	1	3	2
175–	0	2	2
180–185	0	0	1

a) Construct a cumulative frequency table for each year.

b) Draw the cumulative frequency curve for each year.

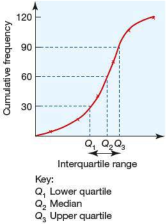
c) Show how your graph could be used to find the median height for each year.

d) What does the median value tell us?

Quartiles and the inter-quartile range

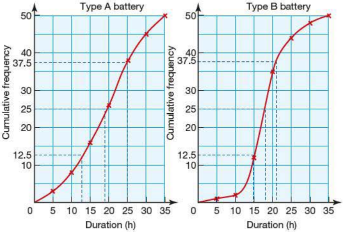
The cumulative frequency axis can also be represented in terms of **percentiles**. A percentile scale divides the cumulative frequency scale into hundredths. The maximum value of cumulative frequency is found at the 100th percentile. Similarly the median, being the middle value, is called the 50th percentile. The 25th percentile is known as the lower quartile, and the 75th percentile is called the upper quartile as introduced in Section 5.

The range of a distribution is found by subtracting the lowest value from the highest value. Sometimes this will give a useful result, but often it will not. A better measure of spread is given by looking at the spread of the middle half of the results, i.e. the difference between the upper and lower quartiles, i.e. the inter-quartile range.



Worked example Consider again the two types of batteries A and B discussed earlier (page 458).

a) Using the graphs, estimate the upper and lower quartiles for each battery.



Lower quartile of Type A ≈ 13 h
Upper quartile of Type A ≈ 25 h
Lower quartile of Type B ≈ 15 h
Upper quartile of Type B ≈ 21 h

b) Calculate the inter-quartile range for each type of battery.

Inter-quartile range of type A ≈ 12 h
Inter-quartile range of type B ≈ 6 h

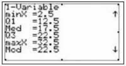
c) Based on these results, how might the manufacturers advertise the two types of batteries?

Type A: on 'average' the longer-lasting battery
Type B: the more reliable battery

The inter-quartile range can be calculated using a graphics calculator:

Casio

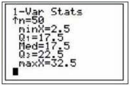
The screen opposite is the result of performing statistical calculations on the Battery A data above. The inter-quartile range can be calculated as a result: IQR $= Q_3 - Q_1 = 22.5 - 12.5 = 10$.



```
1-Variable
minX =2.5
Q1   =12.5
Med  =17.5
Q3   =22.5
maxX =32.5
Mod  =22.5
```

Texas

The screen opposite is the result of performing statistical calculations on the Battery B data above. The inter-quartile range can be calculated as a result: IQR $= Q_3 - Q_1 = 22.5 - 17.5 = 5$.



```
1-Var Stats
↑n=50
 minX=2.5
 Q1=17.5
 Med=17.5
 Q3=22.5
 maxX=32.5
■
```

Exercise 11.12

1. Using the results obtained from Q.2 of Exercise 11.11:
   a) find the inter-quartile range of each of the classes taking the Mathematics test,
   b) analyse your results and write a short summary comparing the three classes.

2. Using the results obtained from Q.3 of Exercise 11.11:
   a) find the inter-quartile range of the students' heights each year
   b) analyse your results and write a short summary comparing the three years.

3. Forty boys enter for a school javelin competition. The distances thrown are recorded below:

Distance thrown (m)	0–	20–	40–	60–	80–100
Frequency	4	9	15	10	2

   a) Construct a cumulative frequency table for the above results.
   b) Draw a cumulative frequency curve.
   c) If the top 20% of boys are considered for the final, estimate (using the graph) the qualifying distance.
   d) Calculate the inter-quartile range of the throws.
   e) Calculate the median distance thrown.

4. The masses of two different types of oranges are compared. Eighty oranges are randomly selected from each type and weighed. The results are shown below:

Type A	
Mass (g)	**Frequency**
75–	4
100–	7
125–	15
150–	32
175–	14
200–	6
225–250	2

Type B	
Mass (g)	**Frequency**
75–	0
100–	16
125–	43
150–	10
175–	7
200–	4
225–250	0

a) Construct a cumulative frequency table for each type of orange.

b) Draw a cumulative frequency graph for each type of orange.

c) Calculate the median mass for each type of orange.

d) Using your graphs estimate:

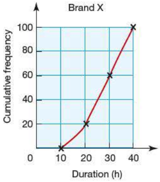
   i) the lower quartile

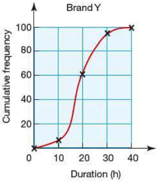
   ii) the upper quartile

   iii) the inter-quartile range for each type of orange.

e) Write a brief report comparing the two types of oranges.

5. Two competing brands of batteries are compared. One hundred batteries of each brand are tested and the duration of each is recorded. The results of the tests are shown in the cumulative frequency graphs below.





a) The manufacturers of brand X claim that on average their batteries will last at least 40% longer than those of brand Y. Showing your method clearly, decide whether this claim is true.

b) The manufacturers of brand X also claim that their batteries are more reliable than those of brand Y. Is this claim true? Show your working clearly.

SECTION 7 Scatter diagrams, correlation and lines of best fit

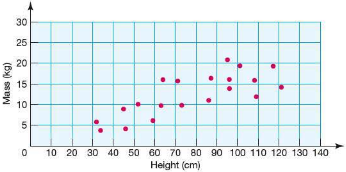
When information about two different aspects (**variables**) of a data item is recorded, such as height and mass of children, we are collecting **bivariate data**. We can use the values of the two variables as the coordinates of a point to represent it on a **scatter diagram** (or scatter graph).

Scatter diagrams are particularly useful if we wish to see if there is a correlation (relationship) between the two variables. How the points lie when plotted indicates the type of relationship between the two sets of data.

Worked example The heights and weights (masses) of 20 children under the age of five were recorded. The heights were recorded in centimetres and the weights in kilograms. The data is shown below with the heights written in red and the weights in blue.

Height	32	34	45	46	52	59	63	64	71	73
Mass	5.834	3.792	9.037	4.225	10.149	6.188	9.891	16.010	15.806	9.929
Height	86	87	95	96	96	101	108	109	117	121
Mass	11.132	16.443	20.895	16.181	14.000	19.459	15.928	12.047	19.423	14.331

i) Plot a scatter diagram for this data.



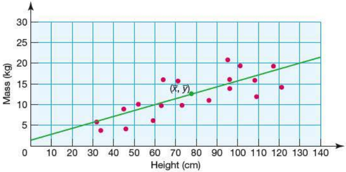
ii) Comment on any relationship that you see.

The points tend to lie in a diagonal direction from bottom left to top right. This suggests that as height increases then, in general, weight increases too. Therefore there is a positive correlation between height and weight.

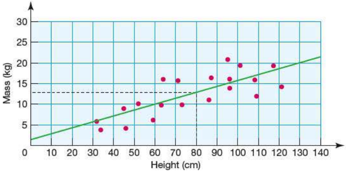
iii) Estimate the weight of another child with a height of 80 cm.

We have to assume that this child will follow the trend set by the other 20 children. To find an approximate value for the weight, we draw a line of best fit. This is a solid straight line which best passes through the points. It also passes through the point $(\bar{x}, \bar{y})$ as shown below.

Note: $\bar{x}$ and $\bar{y}$ are the means of the x and y values respectively, in this example (77.75, 12.535). A line of best fit need not pass through the origin.



The line of best fit can now be used to give an approximate solution to the question. If a child has a height of 80 cm, you would expect his/her weight, by reading from the graph below, to be in the region of 13 kg.



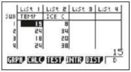
Your graphics calculator will plot scatter diagrams and analyse them. For example, plot the following data for an ice cream vendor on a scatter diagram and, if appropriate, draw a line of best fit.

Temperature (°C)	15	24	18	24	19	26	22	24	27	28	30	25	22	17
Number of ice creams sold	8	34	20	38	28	37	32	29	33	35	44	28	30	25

Casio

MENU 2 to select the stat. mode.

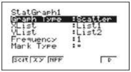
Enter the temperature data in List 1 and the number of ice creams sold in List 2.



```
     List 1 | List 2 | List 3 | List 4
SUB  TEMP   | ICE C  |        |
 1       15 |      8 |        |
 2       24 |     34 |        |
 3       18 |     20 |        |
 4       24 |     38 |        |
                                 15
```

F1 to access the statistical graphing menu.

F6 to check the setup.

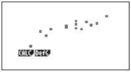


```
StatGraph1
Graph Type    :Scatter
XList         :List1
YList         :List2
Frequency     :1
Mark Type     :▪
```

The graph type is 'scatter' with the *x* values from List 1 and the *y* values from List 2. Each data value is to be counted once.

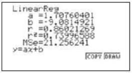
EXE

F1 to plot the scatter diagram.



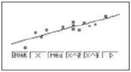
F1 to select the graph calculation menu.

F2 as the line of best fit required is linear. The following screen summarises the properties of the line of best fit in the form $y = ax + b$.



```
LinearReg
   a =1.70760401
   b =-9.0814921
   r =0.86021269
   r²=0.73996588
  MSe=21.256241
y=ax+b
```

F6 to plot the line of best fit.

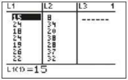


Note: The screen which gives the properties of the line of best fit also gives the value of *r*. This is an indicator of how tight the data is to the line of best fit. It is however beyond the scope of this book.

Texas

to enter the data into lists.

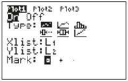
Enter the temperature data in List 1 and the number of ice creams sold in List 2.



```
L1    L2    L3    1
15    8     ------
24    34
18    20
24    38
19    28
26    37
22    32
L1(1)=15
```

to enter the statistical plot setup.

Turn 'Plot 1' to 'On'. Choose the scatter graph and ensure the *x* values are from List 1 and the *y* values from List 2.

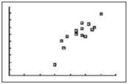


```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Ylist:L2
Mark: □ + ·
```

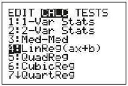
to set scale for each axis.

```
WINDOW
 Xmin=0
 Xmax=35
 Xscl=5
 Ymin=0
 Ymax=50
 Yscl=5
 Xres=1
```

to plot the scatter diagram.



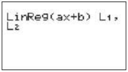
to select the 'Calc' menu.



```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7↓QuartReg
```

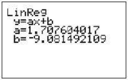
Select  to find the linear equation of the line of best fit through the points.

to calculate the equation of the line of best fit with *x* values from List 1 and *y* values from List 2.



```
LinReg(ax+b) L1,
L2
```

to display the equation in the form $y = ax + b$.

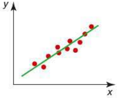


```
LinReg
 y=ax+b
 a=1.707604017
 b=-9.081492109
```

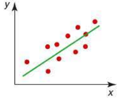
Types of correlation

There are several types of correlation depending on the arrangement of the points plotted on the scatter diagram. These are described below.

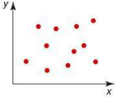
A **strong positive correlation.**
The points lie tightly around the line of best fit. As x increases, so does y.



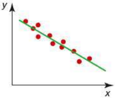
A **weak positive correlation.**
Although there is direction to the way the points are lying, they are not tightly packed around the line of best fit. As x increases, y tends to increases too.



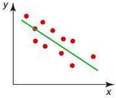
No correlation.
There is no pattern to the way in which the points are lying, i.e. there is no correlation between the variables x and y. As a result, there can be no line of best fit.



A **strong negative correlation.**
The points lie tightly around the line of best fit. As x increases, y decreases.



A **weak negative correlation.**
The points are not tightly packed around the line of best fit. As x increases, y tends to decrease.



Exercise 11.13

1. State what type of correlation you might expect, if any, if the following data was collected and plotted on a scatter diagram. Give reasons for your answer.
   a) A student's score in a Mathematics exam and their score in a Science exam
   b) A student's hair colour and the distance they have to travel to school

c) The outdoor temperature and the number of cold drinks sold by a shop
d) The age of a motorcycle and its second-hand selling price
e) The number of people living in a house and the number of rooms the house has
f) The number of goals your opponents score and the number of times you win
g) A person's height and the person's age
h) A car's engine size and its fuel consumption

2. The table shows the readings for the number of hours of sunshine and the amount of rainfall in millimetres for several cities and towns in the UK.

Place	Hours of sunshine	Rainfall (mm)
Athens	12	6
Belgrade	10	61
Copenhagen	8	71
Dubrovnik	12	26
Edinburgh	5	83
Frankfurt	7	70
Geneva	10	64
Helsinki	9	68
Innsbruck	7	134
Krakow	7	111
Lisbon	12	3
Marseilles	11	11
Naples	10	19
Oslo	7	82
Plovdiv	11	37
Reykjavik	6	50
Sofia	10	68
Tallinn	10	68
Valletta	12	0
York	6	62
Zurich	8	136

a) Plot a scatter diagram of hours of sunshine against amount of rainfall. Use a spreadsheet or graphing software if possible.
b) What type of correlation, if any, is there between the two variables? Comment on whether this is what you would expect.

3. The United Nations keeps an up-to-date database of statistical information on its member countries. The table below shows some of the information available.

Country	Life expectancy at birth (years, 2005–10)		Adult illiteracy rate (%, 2007) Total	Infant mortality rate (per 1,000 births, 2005–10)
	Female	Male		
Australia	84	79	0	5
Barbados	80	74	2	10
Brazil	76	69	10	23
Chad	50	47	68	130
China	75	71	7	23
Colombia	77	69	7	19
Congo	55	53	26	79
Cuba	81	77	0	5
Egypt	72	68	34	35
France	85	78	0	4
Germany	82	77	0	4
India	65	62	34	55
Iraq	72	63	26	33
Israel	83	79	4	5
Japan	86	79	0	3
Kenya	55	54	26	64
Mexico	79	74	7	17
Nepal	67	66	43	42
Portugal	82	75	5	4
Russian Federation	73	60	0	12
Saudi Arabia	75	71	15	19
United Kingdom	82	77	0	5
United States of America	81	77	0	5

a) By plotting a scatter diagram, decide if there is a correlation between the Adult illiteracy rate and the Infant mortality rate.

b) Are your findings in part (a) what you expected? Explain your answer.

c) Without plotting a scatter diagram decide if you think there is likely to be a correlation between male and female life expectancy at birth. Explain your reasons.
d) Plot a scatter diagram to test if your predictions in part (c) were correct.

4. The table below gives the average time taken for 30 students in a class to get to school each morning and the distance they live from the school.

Distance (km)	2	10	18	15	3	4	6	2	25	23	3	5	7	8	2
Time (mins)	5	17	32	38	8	14	15	7	31	37	5	18	13	15	8
Distance (km)	19	15	11	9	2	3	4	3	14	14	4	12	12	7	1
Time (mins)	27	40	23	30	10	10	8	9	15	23	9	20	27	18	4

a) Plot a scatter diagram of distance travelled against time taken.
b) Describe the correlation between the two variables.
c) Explain why some students who live further away, may get to school quicker than some of those who live nearer.
d) Draw a line of best fit on your scatter diagram.
e) A new student joins the class. Use your line of best fit to estimate how far away she might live if she takes, on average, 19 minutes to get to school each morning.

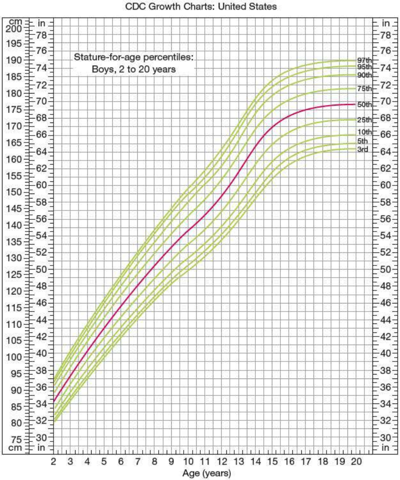
5. A department store decides to investigate if there is a correlation between the number of pairs of gloves it sells and the outside temperature. Over a one-year period it records, every two weeks, how many pairs of gloves are sold and the mean daytime temperature during the same period. The results are given in the table below:

Mean temp (°C)	3	6	8	10	10	11	12	14	16	16	17	18	18
Number of pairs of gloves	61	52	49	54	52	48	44	40	51	39	31	43	35
Mean temp (°C)	19	19	20	21	22	22	24	25	25	26	26	27	28
Number of pairs of gloves	26	17	36	26	46	40	30	25	11	7	3	2	0

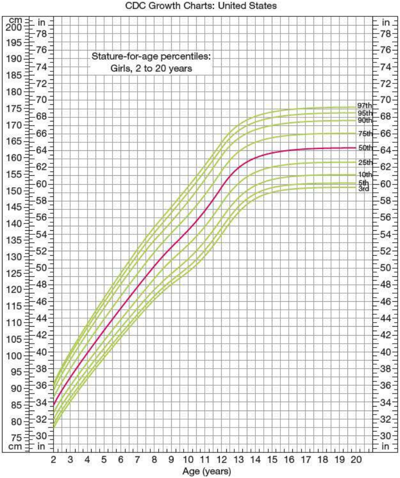
a) Plot a scatter diagram of mean temperature against number of pairs of gloves.
b) What type of correlation is there between the two variables?
c) How might this information be useful for the department store in the future?

Heights and percentiles

The graphs below show the height charts for males and females from the age of 2 to 20 years in the United States.



Note: Heights have been given in both centimetres and inches.



1. From the graph find the height corresponding to the 75th percentile for 16 year-old girls.
2. Find the height which 75% of 16 year-old boys are taller than.
3. What is the median height for 12 year-old girls?
4. Measure the heights of students in your class. By carrying out appropriate statistical calculations, write a report comparing your data to that shown in the graphs.
5. Would all cultures use the same height charts? Explain your answer.

Reading ages

Depending on their target audience, newspapers/magazines and books have different levels of readability. Some are easy to read and others more difficult.

1. Decide on some factors that you think would affect the readability of a text.
2. Write down the names of two newspapers which you think would have different reading ages. Give reasons for your answer.

There are established formulae for calculating the reading age of different texts.

One of these is the Gunning Fog Index. It calculates the reading age as follows:

Reading age $= \frac{2}{5}\left(\frac{A}{n} + \frac{100L}{A}\right)$ where

A = number of words
n = number of sentences
L = number of words with 3 or more syllables

3. Choose one article from each of the two newspapers you chose in Q.2. Use the Gunning Fog Index to calculate the reading ages for the articles. Do the results support your predictions?
4. Write down some factors which you think may affect the reliability of your results.

ICT Activity

In this activity you will be collecting the height data of all the students in your class and plotting a cumulative frequency graph of the results.

1. Measure the heights of all the students in your class.
2. Group your data appropriately.
3. Enter your data into graphing software such as Excel or Autograph.
4. Produce a cumulative frequency graph of the results.
5. From your graph find:
   a) the median height of the students in your class.
   b) the inter-quartile range of the heights.
6. Compare the cumulative frequency graph from your class with one produced from data collected from another class in a different year group. Comment on any differences/similarities between the two.

Student assessment 1

1. Find the mean, median and mode of the following sets of data:

a) 63 72 72 84 86
b) 6 6 6 12 18 24
c) 5 10 5 15 5 20 5 25 15 10

2. The mean mass of the 15 players in a rugby team is 85 kg. The mean mass of the team plus a substitute is 83.5 kg. Calculate the mass of the substitute.

3. Thirty families live in a street. The number of children in each family is given in the table below:

Number of children	0	1	2	3	4	5	6
Frequency	3	5	8	9	3	0	2

a) Calculate the mean number of children per family.
b) Calculate the median number of children per family.
c) Calculate the modal number of children per family.

4. The number of people attending thirty screenings of a film at a local cinema are given below:

21 30 66 71 10 37 24 21 62 50 27 31 65 12 38
34 53 34 19 43 70 34 27 28 52 57 45 25 30 39

a) Using groups 10–19, 20–29, 30–39, etc, present the above data in a grouped frequency table.
b) Using your grouped data, calculate an estimate for the mean number of people attending each screening.

5. Identify which of the following types of data are discrete and which are continuous:

a) The number of cars passing the school gate each hour
b) The time taken to travel to school each morning
c) The speed at which students run in a race
d) The wingspan of butterflies
e) The height of buildings

6. A businesswoman travels to work in her car each morning in one of two ways; either using the small country roads or using the motorway. She records the time taken to travel to work each day. The results are shown in the table below:

Time (min)	$10 \leqslant t < 15$	$15 \leqslant t < 20$	$20 \leqslant t < 25$	$25 \leqslant t < 30$	$30 \leqslant t < 35$	$35 \leqslant t < 40$	$40 \leqslant t < 45$
Motorway frequency	3	5	7	2	1	1	1
Country roads frequency	0	0	9	10	1	0	0

a) Complete a cumulative frequency table for each of the sets of results shown above.

b) Using your cumulative frequency tables, plot two cumulative frequency curves – one for the time taken to travel to work using the motorway, the other for the time taken to travel to work using country lanes.

c) Use your graphs to work out the following for each method of travel:

i) the median travelling time
ii) the upper and lower quartile travelling times
iii) the inter-quartile range for the travelling times

d) With reference to your graphs or calculations, explain which is the most reliable way of getting to work.

e) If she had to get to work one morning within 25 minutes of leaving home, which way would you recommend she take. Explain your answer fully.

7. Twenty students take three long jumps. The best result for each student (in metres) is recorded below:

4.3	5.4	4.3	4.0	3.8	5.1	3.6	5.5	6.2	4.7
5.2	3.8	2.4	4.7	3.9	5.6	5.8	4.7	3.3	2.9

The students were then coached in long jump technique and given three further jumps. Their individual best results are recorded below:

4.7	5.9	4.8	4.6	4.5	5.3	5.2	5.5	6.3	4.9
5.2	4.9	5.6	5.3	6.8	5.4	5.8	5.4	4.3	5.5

Draw a back-to-back stem-and-leaf diagram of their long jumps before and after coaching.

Comment on your diagram.

8. The popularity of a group of professional football players and their yearly salary is given in the table below:

Popularity	1	2	3	4	5	6	7	8	9	10
Salary ($ million)	4.8	3.6	4.5	3.1	7.7	6.3	2.9	3.1	4.1	1.8
Popularity	11	12	13	14	15	16	17	18	19	20
Salary ($ million)	4.5	3.1	2.7	3.9	6.2	5.8	4.1	5.3	7.2	6.5

a) Using your graphics calculator, calculate the equation of the line of best fit.

b) The statement is made in a newspaper 'Big money footballers are not popular with fans'. Comment on this statement in the light of your answer to part (a).

Student assessment 2

1. Find the mean, median and mode of the following sets of data:

a) 4 5 5 6 7

b) 3 8 12 18 18 24

c) 4 9 3 8 7 11 3 5 3 8

2. The mean mass of the 11 players in a football team is 76 kg. The mean mass of the team plus a substitute is 76.2 kg. Calculate the mass of the substitute.

3. Thirty children were asked about the number of pets they had. The results are shown in the table below.

Number of pets	0	1	2	3	4	5	6
Frequency	5	5	3	7	3	1	6

a) Calculate the mean number of pets per child.

b) Calculate the median number of pets per child.

c) Calculate the modal number of pets.

4. The number of people attending a disco at a club's over 30s evenings are:

89 94 32 45 57 68 127 138 23 77 99 47 44 100 106
132 28 56 59 49 96 103 90 84 136 38 72 47 58 110

a) Using groups 0–19, 20–39, 40–59 etc, present the above data in a grouped frequency table.

b) Using your grouped data, calculate an estimate for the mean number of people going to the disco each night.

5. Identify which of the following types of data are discrete and which are continuous:

a) The number of goals scored in a hockey match

b) Dress sizes

c) The time taken to fly from Hong Kong to Beijing

d) The price of a kilogram of carrots

e) The speed of a police car

6. Four hundred students sit their IGCSE Mathematics exam. Their marks (as percentages) are shown in the table below:

Mark (%)	Frequency	Cumulative frequency
31–40	21	
41–50	55	
51–60	125	
61–70	74	
71–80	52	
81–90	45	
91–100	28	

a) Copy and complete the above table by calculating the cumulative frequency.
b) Draw a cumulative frequency curve of the results.
c) Using the graph, estimate a value for:
  i) the median exam mark
  ii) the upper and lower quartiles
  iii) the inter-quartile range.

7. Eight hundred students sit an exam. Their marks (as percentages) are shown in the table below:

Mark (%)	Frequency	Cumulative frequency
1–10	10	
11–20	30	
21–30	40	
31–40	50	
41–50	70	
51–60	100	
61–70	240	
71–80	160	
81–90	70	
91–100	30	

a) Copy and complete the above table by calculating the cumulative frequency.
b) Draw a cumulative frequency curve of the results.
c) An A grade is awarded to any student achieving at or above the upper quartile. Using your graph, identify the minimum mark required for an A grade.
d) Any student below the lower quartile is considered to have failed the exam. Using your graph, identify the minimum mark needed so as not to fail the exam.
e) How many students failed the exam?
f) How many students achieved an A grade?

Sample examination papers

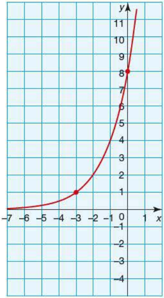
Sample examination Paper 2 (Extended)

Non calculator **40 marks**

1. Write down the value of:
a) 2^{-3} [1]
b) $125^{\frac{1}{3}}$ [1]

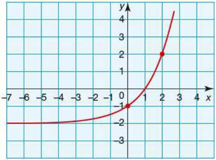
2. The graphs shown are translations of the graph $y = 2^x$.
Write down their equations.

a)



[2]

b)



[2]

3. For the sequence 6, 10, 14, 18, 22, …
a) find the next two terms [1]
b) find a formula for the *n*th term. [2]

4. Solve the simultaneous equations.
$2x = 18 - 4y$
$3x - 3y = 9$ [3]

5. Solve the equation $3x^2 = 8 - 2x$. [4]

6. a) Write down the value of $\log_9 27$. [2]
b) Evaluate $\log_5 50 + \log_5 \frac{5}{2}$. [3]

7. Simplify:

a) $\sqrt{72}$ [1]

b) $\sqrt{72} \times \sqrt{8}$ [2]

c) $\dfrac{\sqrt{72}}{\sqrt{18}}$ [2]

8. For the set of data

1 0 2 4 1 2 1 1 2 5 5 0 1 2 3

find:

a) the mean [1]

b) the median [1]

c) the mode. [1]

9. A right-angled triangle ABC has hypotenuse AC 10 cm and side AB 8 cm. The angle C is $x°$.

a) Calculate the length of the third side BC. [2]

b) Find:

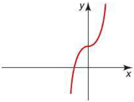
- i) $\sin x°$ [1]
- ii) $\cos x°$ [1]
- iii) $\tan x°$. [1]

10. The graphs (**a**) to (**f**) below show some of the following functions (A to H).

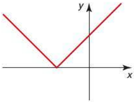
A $f(x) = \cos x + 1$
B $f(x) = 2 \sin x$
C $f(x) = (x - 2)^3$
D $f(x) = x^3 + 2$
E $f(x) = -x^2 + 4$
F $f(x) = -x^2 + 2x + 4$
G $f(x) = |x + 3|$
H $f(x) = |x - 3|$

Make each graph with its correct function.

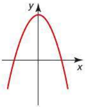
a)



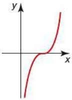
b)



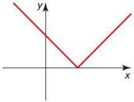
c)



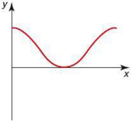
d)



e)



f)



[6]

Sample examination Paper 4 (Extended)

120 marks

1. Two trains depart at the same time from cities M and N, which are 200 km apart. Train A travels from M to N, train B from N to M. Train A travels a distance of 60 km in the first hour, 120 km in the next
1.5 hours and the rest of its journey at 40 km/h.
Train B departing from N travels the whole distance at a speed of 100 km/h.

a) How long does it take train A to complete the journey? [3]

b) What is the average speed of train A? [2]

c) Draw a distance–time graph to show both journeys. [4]

d) Estimate from your graph the distance from city M of the trains when they pass each other. [2]

e) Estimate how long after the start of the journey it is when the trains pass each other. [2]

2. a) Show that $x = 1 + \dfrac{7}{x-5}$ can be written as $x^2 - 6x - 2 = 0$. [2]

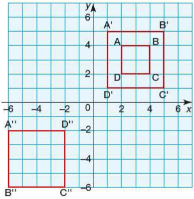
b) Use the quadratic formula to solve the equation $x = 1 + \dfrac{7}{x-5}$.
Leave your answer(s) in surd form. [4]

c) Sketch the graph of this function, showing clearly where it crosses both axes. [3]

3. a) The square ABCD is mapped onto A'B'C'D'.
A'B'C'D' is then mapped onto A"B"C"D".

i) Describe fully the transformation which maps ABCD onto A'B'C'D'. [2]

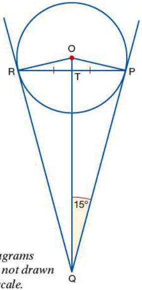
ii) Describe fully the transformation which maps A'B'C'D' onto A"B"C"D". [2]



b) $\mathbf{p} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

i) Calculate the magnitude of the vector $4\mathbf{p} - \mathbf{r}$, giving your answer to one decimal place. [4]

ii) If $a\mathbf{p} + b\mathbf{q} = \begin{pmatrix} -2 \\ 14 \end{pmatrix}$, find the values of a and b. [4]



4. In the diagram (left) O marks the centre of the circle. PQ and RQ are tangents to the circle.

$\angle PQO$ is 15°

a) Calculate:
   i) $\angle POQ$ [2]
   ii) $\angle TRO$. [2]

b) If OP = 6 cm, calculate the length of the chord PR. [3]

Diagrams are not drawn to scale.

5. $f(x) = \dfrac{2x + 6}{3}$

a) Find $f(\frac{3}{2})$. [2]

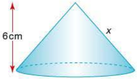
b) Find $f(-1)$. [2]

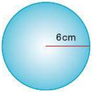
c) Find $f^{-1}(2)$. [3]

6. A is the point (1, –1) and B is the point (4, 8).

a) Find the equation of the straight line which passes through A and B. [3]

b) Find the equation of the line perpendicular to AB which passes through the mid-point of AB. Give your answer in the form $ax + by = d$ where a, b and d are integers. [5]





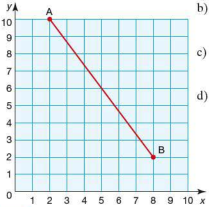
7. The cone and the sphere shown (left) have the same volume. If the radius of the sphere and the height of the cone are both 6 cm, calculate:

a) the volume of the cone [2]

b) the base radius of the cone [3]

c) the slant height x cm [3]

d) the total surface area of the cone. [4]



8. Points A and B have coordinates (2, 10) and (8, 2) respectively.

a) Calculate the length of the line segment AB. [2]

b) Triangle ABC is a right-angled triangle, where BC is parallel to the x-axis. What is the coordinate of point C? [2]

c) A, B and C all fall on the circumference of a circle. What are the coordinates of the centre of the circle O? [2]

d) Calculate the area of the circle, leaving your answer in terms of π. [3]

9. a) Use your calculator to sketch the graph of $y = x^3 - 2x^2 - 5x + 6$. [3]

b) Showing your method clearly, **calculate** where the graph crosses the y-axis. Mark this value on your graph. [2]

c) Use your calculator to find the roots of the equation and mark these on your graph. [3]

d) Hence, using your solutions to **part (c)** write down the equation of the graph as a product of three linear factors. [3]

10. In a class of 30 students, 18 study English Language, 25 study English Literature and 3 study neither. x is the number of students who study both subjects.

a) Show this information on a Venn diagram. [3]

b) How many students study both English Language and English Literature? [3]

11. Twelve athletes take part in a race which consists of a 10 km run and a 25 km cycle ride. The table shows the amount of time taken by each athlete during each section of the race, in minutes.

Athlete	1	2	3	4	5	6	7	8	9	10	11	12
Running time (x) (min)	45	42	36	40	48	32	44	38	51	42	46	40
Cycling time (y) (min)	53	49	44	47	52	41	52	44	60	48	54	46

a) Calculate the mean running time $\bar{x}$. [1]

b) Calculate the mean cycling time $\bar{y}$. [1]

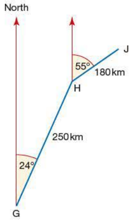
c) Draw a scatter diagram to show the running times and the cycling times of the athletes and also mark on the point $(\bar{x}, \bar{y})$. [3]

d) Draw a line of best fit for the data. [2]

e) Use your calculator to find the equation of the line of best fit. [3]

f) An athlete took 70 minutes to complete the cycle ride. Use your answer to **part (e)** to predict his running time. [2]

g) State what assumption you have made when answering **part (f)**. [1]



12. An aeroplane sets off from position G on a bearing of 024° towards H, 250 km away. At H it changes course and heads towards J on a bearing of 055° and a distance of 180 km away as shown (left).

a) Calculate how far H is to the North of G. [2]

b) Calculate how far H is to the East of G. [2]

c) What is the shortest distance between G and J? [3]

d) Calculate the bearing of G from J. [6]

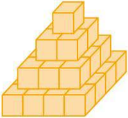
Sample examination Paper 6 (Extended)

40 marks

A. Investigation Pyramids

Different types of pyramids are found all over the world. The most famous are of course the Egyptian Pyramids. These are square-based pyramids made of blocks. A simplified version is shown below.

Assuming each block is 1 × 1 × 1 m cube, this pyramid is 2 m high and is made of 5 blocks.



This pyramid is 4 m high and is made of 30 blocks.

1. Calculate the number of blocks needed for a pyramid with each of these heights.
a) 3 m b) 5 m c) 10 m

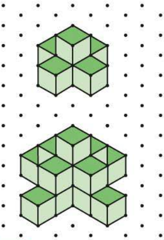
2. Enter your results in an ordered table.

3. Show, by using the method of differences, that the algebraic rule for calculating the total number of blocks (n) needed for a pyramid (h) metres tall is

$$n = \frac{h(h+1)(2h+1)}{6}$$

4. The Great Pyramid at Giza in Egypt is approximately 145 m tall. Assuming it is a square-based pyramid built of 1 × 1 × 1 m blocks and assuming it is a solid construction (i.e. with no internal chambers or corridors), calculate the number blocks needed for its construction.

Ziggurats are another type of pyramid. This style of pyramid originated from Mesopotamia (modern day Iraq). Simplified versions of these are shown (left).
(The diagrams are shown on isometric grid for clarity.)



Assuming each block is 1 × 1 × 1 m cube, this pyramid is 2 m high and is made of 6 blocks.

This Pyramid is 3 m high and is made of 15 blocks.

5. Calculate the number of blocks needed for a pyramid of each of these heights.
   a) 4 m
   b) 5 m
   c) 10 m

6. Enter your results in an ordered table.

7. Find an algebraic rule for calculating the total number of blocks (n) needed for a pyramid (h) metres tall.

8. The Great Ziggurat of Ur in Iraq is approximately 30 m tall. Assuming it is built of 1 × 1 × 1 m blocks calculate the number blocks needed for its construction.

B. Modelling Stopping distance

The stopping distance of a car depends on a number of factors. These include the surface conditions of the road (i.e. whether it is wet or dry), the type of tyres on the car and also the speed at which the car is travelling.

It is estimated that the typical stopping distance for a car travelling at 32 km/h is 12 metres.

1. Show that 32 km/h is equivalent to approximately 9 m/s.

   The table below shows the typical stopping distances (d) in metres of cars travelling at a speed of s m/s.

Speed (m/s)	0	9	13	18	22	27	31
Stopping distance (m)	0	12	23	36	53	73	96

2. a) Plot a scatter diagram with speed along the horizontal axis and stopping distance along the vertical axis.
   b) Assuming that the relationship between the two is quadratic, draw a curve of best fit through the points.
   c) Find the approximate equation of the quadratic curve.

3. a) Use your equation to predict the stopping distance for a car traveling at 40 m/s.
   b) Is the equation reliable for calculating the speed of a car that took 500 m to stop? Justify your answer clearly and with numerical evidence.

Index

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Accompanying material to support this textbook is available online at:

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This material includes PowerPoints, additional student assessments, modelling and investigation questions, sample papers with solutions and Personal Tutors. Our unique Personal Tutor technology provides audio-visual walkthroughs to help your understanding of more complex topics.

You will need the following user name and password to access the online material:

User name: igcseintlmaths
Password: equation

Please note that the user name and password are both case sensitive.